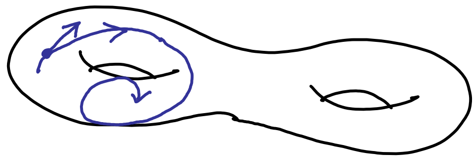
# Lecture 1-2



**Chaotic flows:**

- geodesic flow,
- horocycle flow.

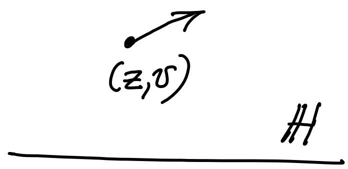
**Eigenstates:**

$\Delta \varphi_\lambda = \lambda \varphi_\lambda$

$\varphi_\lambda \approx ?$ as $\lambda \to \infty$.

## Hyperbolic plane.

$\mathbb{H} = \{ x+iy \in \mathbb{C} : y > 0 \}$




$T\mathbb{H} = \{ (z,v) : z \in \mathbb{H},\ v \in \mathbb{C} \}$

tangent bundle of $\mathbb{H}$.

$$T\mathbb{H} = \bigsqcup_{z \in \mathbb{H}} T_z \mathbb{H}.$$

**Hyperbolic metric:** for $v \in T_z\mathbb{H}$, $\|v\|_z = \dfrac{\|v\|}{\operatorname{Im}(z)}$.

For a paths $\varphi : [0,1] \to \mathbb{H}$,

the hyperbolic length: $L(\varphi) = \int_0^1 \|\varphi'(t)\|_{\varphi(t)}\, dt.$

For $z_1, z_2 \in \mathbb{H}$, $d(z_1,z_2) = \inf_{\varphi} L(\varphi)$

where $\varphi$ runs over continuous piecewise differentiable curves with $\varphi(0)=z_1$, $\varphi(1)=z_2$.

Isometries: for $g=\begin{pmatrix} a & b \\ c & d\end{pmatrix} \in SL_2(\mathbb{R})$,

$$g\cdot z = \frac{az+b}{cz+d}.$$

Since $Im(g\cdot z) = \dfrac{Im(z)}{|cz+d|^2}$, $\quad g\cdot\mathbb{H} \subset \mathbb{H}$.

Lem. $SL_2(\mathbb{R})$ acts transitively on $\mathbb{H}$,
$Stab(i) = SO(2) = \left\{\begin{pmatrix} a & b \\ -b & a\end{pmatrix} : a^2+b^2=1\right\}$.

$$\begin{pmatrix} 1 & x \\ 0 & 1\end{pmatrix}\begin{pmatrix} y^{1/2} & 0 \\ 0 & y^{-1/2}\end{pmatrix}\cdot i = x+iy.$$

Lem. $SL_2(\mathbb{R})$ preserves the hyperbolic metric.

$$L(g\cdot\varphi) = \int_0^1 \|(Dg)_{\varphi(t)}\,\varphi'(t)\|_{g\cdot\varphi(t)}\,dt,$$

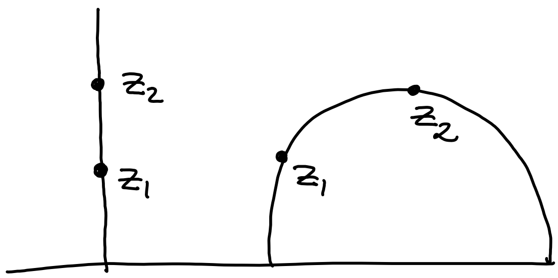
where $(Dg)_z v = \dfrac{v}{(cz+d)^2}$, so

$$L(g\cdot\varphi) = \int_0^1 \frac{\|\varphi'(t)\|_{g\cdot\varphi(t)}}{|c\varphi(t)+d|^2}\,dt = \int_0^1 \frac{\|\varphi'(t)\|\,dt}{Im(g\cdot\varphi(t))\cdot|c\varphi(t)+d|^2}$$

$$= \int_0^1 \frac{\|\varphi'(t)\|}{Im(\varphi(t))}\,dt = L(\varphi).$$

<u>Geodesics:</u> A <u>geodesic</u> between $z_1, z_2 \in \mathbb{H}$ is a path between $z_1$ and $z_2$ such that $L(\varphi) = d(z_1, z_2)$.

<u>Lem.</u> The geodesic between $z_1, z_2 \in \mathbb{H}$ is either:
- vertical lines (if $Re(z_1) = Re(z_2)$),
- semicircle with the center on $x$-axis.




⌈ Suppose that $Re(z_1) = Re(z_2)$.
Given a path $\varphi : [0,1] \to \mathbb{H}$ between $z_1, z_2$,

$$L(\varphi) = \int_0^1 \frac{\sqrt{\varphi_1'(t)^2 + \varphi_2'(t)^2}}{\varphi_2(t)} \geqslant \int_0^1 \frac{|\varphi_1'(t)|}{\varphi_2(t)}\, dt,$$

where "=" $\Leftrightarrow$ $\varphi_1' = 0$
Hence, the shortest path is the vertical line.
In general, we choose $g \in SL_2(\mathbb{R})$ so that

$$Re(g \cdot z_1) = Re(g \cdot z_2).$$

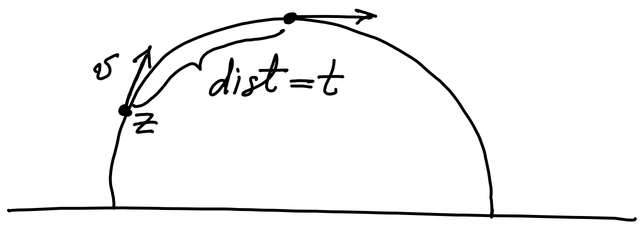
Then the vertical line $L$ is the geodesic between $g \cdot z_1$ and $g \cdot z_2$, and $g^{-1} L$, which is a semi-circle, is the geodesic between $z_1$ and $z_2$.

## Geodesic flow:

$$T^1 \mathbb{H} = \{(z, v) : z \in \mathbb{H},\ v \in T_z \mathbb{H} : \|v\|_z = 1\}$$

unit tangent bundle

$g_t : T^1 \mathbb{H} \to T^1 \mathbb{H}$ - geodesic flow




$SL_2(\mathbb{R})$ acts on $T^1 \mathbb{H}$:

$$(z, v) \xmapsto{g} (g \cdot z, (Dg)_z v) = \left(g \cdot z, \frac{v}{(cz+d)^2}\right).$$

$$\mathrm{Stab}((i, i)) = \{\pm I\}.$$

Hence, $T^1 \mathbb{H} \simeq SL_2(\mathbb{R}) / \{\pm I\} = PSL_2(\mathbb{R})$.

Note that $d(i, a\cdot i) = \log a$ for $a > 1$,
(check)

so that $g_t \cdot (i,i) = (e^{t}\cdot i, e^{t}\cdot i)$
$$= \begin{pmatrix} e^{t/2} & 0 \\ 0 & e^{-t/2} \end{pmatrix} \cdot (i,i)$$

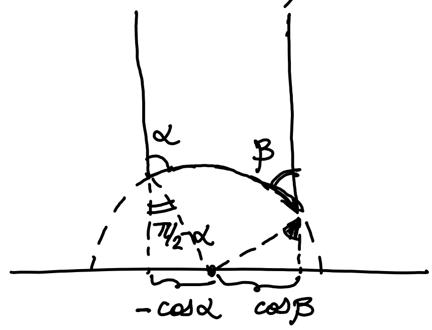
Using the identification $T^1\mathbb{H} \simeq PSL_2(\mathbb{R})$,

$$g_t : PSL_2(\mathbb{R}) \longrightarrow PSL_2(\mathbb{R})$$
$$g \longmapsto g \cdot \begin{pmatrix} e^{t/2} & 0 \\ 0 & e^{-t/2} \end{pmatrix}.$$

Hyperbolic area: $\frac{dx\,dy}{y^2}$ is invariant under $SL_2(\mathbb{R})$.

Lem. If $T$ is a hyperbolic triangle with angles $\alpha, \beta, \gamma$, then $|T| = \pi - \alpha - \beta - \gamma$.

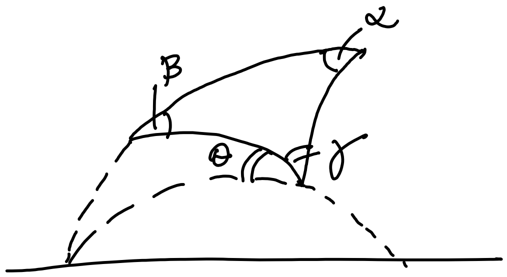
Suppose that $T$ has one of the vertices in $\mathbb{R} \cup \{\infty\}$. Applying $g \in SL_2(\mathbb{R})$, we may assume that $|v_1| = |v_2| = 1$, $v_3 = \infty$. Then




$$|T| = \int_{\cos(\pi-\alpha)}^{\cos\beta} \int_{\sqrt{1-x^2}}^{\infty} \frac{dx\,dy}{y^2}$$

$$= \pi - \alpha - \beta.$$

In general, we use the picture




$$|T| = \pi - \alpha - (\gamma + \theta) - (\pi - (\pi - \beta) - \theta)$$
$$= \pi - \alpha - \beta - \gamma.$$

Cor. If $D$ is hyperbolic $n$-gon with angles $\alpha_1, \dots \alpha_n$, then $|D| = (n-2)\pi - \alpha_1 - \dots - \alpha_n$.

## Hyperbolic surfaces.

Let $\Gamma$ be a discrete subgroup of $SL_2(\mathbb{R})$. We consider the space $X = \Gamma \backslash \mathbb{H}^2$.

Def $F \subset \mathbb{H}^2$ is a fundamental domain if

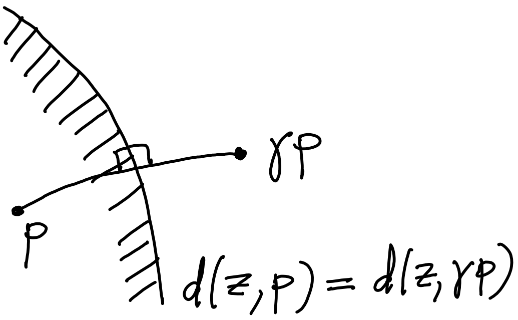
1) $\bigcup_{\gamma \in \Gamma} \gamma F = \mathbb{H}^2$;

2) $\gamma_1 F^\circ \cap \gamma_2 F^\circ = \emptyset$ for $\gamma_1 \neq \gamma_2 \in \Gamma$.

## Dirichlet fundamental domain:

Fix a point $p \in \mathbb{H}$, not fixed by $\gamma \in \Gamma \setminus \{e\}$.

$$D = \{ z \in \mathbb{H} : d(z,p) < d(z, \gamma p) \text{ for } \gamma \in \Gamma \setminus \{e\} \}.$$




D is convex

Prop. $D$ is a fundamental domain.

Lem. The map $SL_2(\mathbb{R}) \to \mathbb{H} : g \mapsto g z_0$ is proper.

$\ulcorner$ Without loss of generality, $z_0 = i$.

$$g = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} y^{1/2} & 0 \\ 0 & y^{-1/2} \end{pmatrix} \underset{SO(2)}{\underset{\cap}{k}} \longmapsto x + iy$$ $\lrcorner$

Proof. By Lem., for every $z_0 \in \mathbb{H}$, the orbit $\Gamma z_0$ is discrete. In particular,

$$\exists\, z \in \Gamma z_0 : \quad d(z,p) \le d(\gamma z_0, p) = d(z_0, \gamma^{-1} p)$$
for all $\gamma \in \Gamma$.

Then the geodesic segment $[p, z) \subset D$.

Hence, $z \in \overline{D}$, and $\Gamma \cdot \overline{D} = \mathbb{H}$.

Suppose that for $z_1, z_2 \in D$, $z_1 = \gamma z_2$ with $\gamma \neq e$.

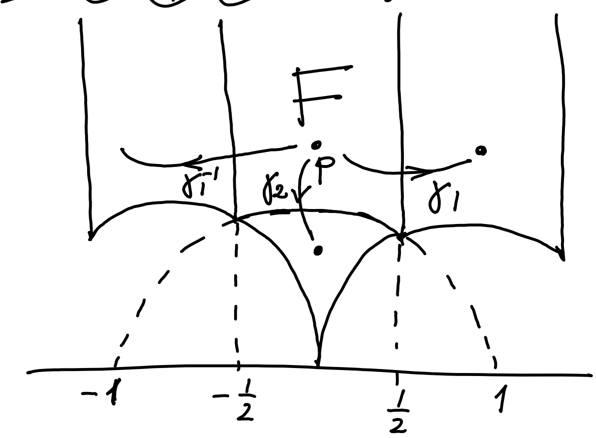
Then $d(z_1, p) < d(z_1, \gamma p) = d(\underbrace{\gamma^{-1} z_1}_{z_2}, p)$,

and similarly, $d(z_2, p) < d(z_1, p)$,

with is a contradiction.

Hence, $D \cap \gamma D = \emptyset$ for $\gamma \neq e$. $\square$

Example: 1) $\Gamma = SL_2(\mathbb{Z})$, $p = 2i$.




$\gamma_1 = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$, $\gamma_2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \in \Gamma$.

It is clear that

$D \subset \underbrace{\{ |\mathrm{Re}(z)| < \tfrac{1}{2},\ |z| > 1 \}}_{F}$.

Suppose that $D \subsetneq F$.

Then $\exists z \in F$, $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$, $\gamma \neq e$ : $\gamma \cdot z \in F$.

We have $\mathrm{Im}(\gamma z) = \dfrac{\mathrm{Im}(z)}{|cz+d|^2}$

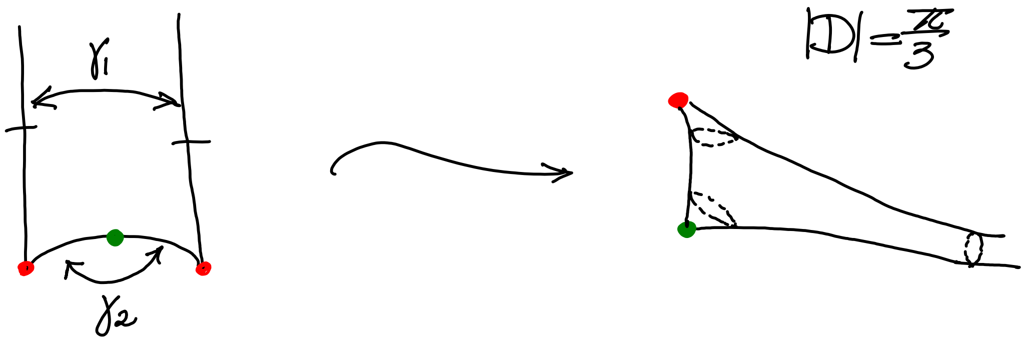
Since $|z|>1$ and $|Re(z)|<\frac{1}{2}$,

$$|cz+d|^2 = c^2|z|^2 + 2cd\,Re(z) + d^2 > c^2+d^2-|cd|$$
$$= (|c|-|d|)^2+|cd|$$

Since $c,d \in \mathbb{Z}$ and $(c,d)\neq(0,0)$, $|cz+d|>1$.

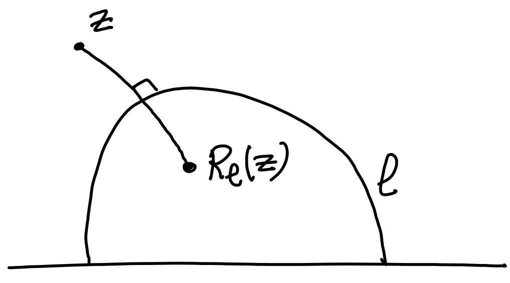
Hence, $Im(\gamma z)<Im(z)$.

Applying the same argument to $\gamma z \xrightarrow{\gamma^{-1}} z$, we obtain $Im(z)<Im(\gamma z)$ — contradiction.




2) <u>Triangle group.</u>

For a geodesic $\ell$, define the reflection map $z \longmapsto R_\ell(z)$.



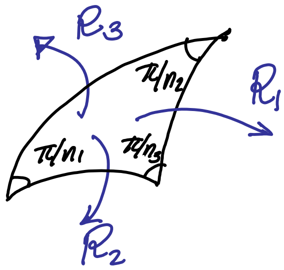

If $\ell_0 = y$-axis, $R_{\ell_0}(z) = -\bar{z}$.

In general take $g \in SL_2(\mathbb{R})$ such that $g\cdot\ell = \ell_0$. Then $R_\ell = g^{-1}\circ R_{\ell_0}\circ g$.

We note that:

- $R_\ell$ preserves the hyperbolic metric,
- $\langle PSL_2(\mathbb{R}), R_\ell : \ell\text{-geodesic} \rangle$ is index 2 supergroup of $PSL_2(\mathbb{R})$.

Take a triangle $T$ with angles $\frac{\pi}{n_1}, \frac{\pi}{n_2}, \frac{\pi}{n_3}$, $n_i \in \mathbb{N}$.
(check: for every $\alpha, \beta, \gamma$ with $\alpha+\beta+\gamma<\pi$, there exists a triangle with angles $\alpha, \beta, \gamma$)



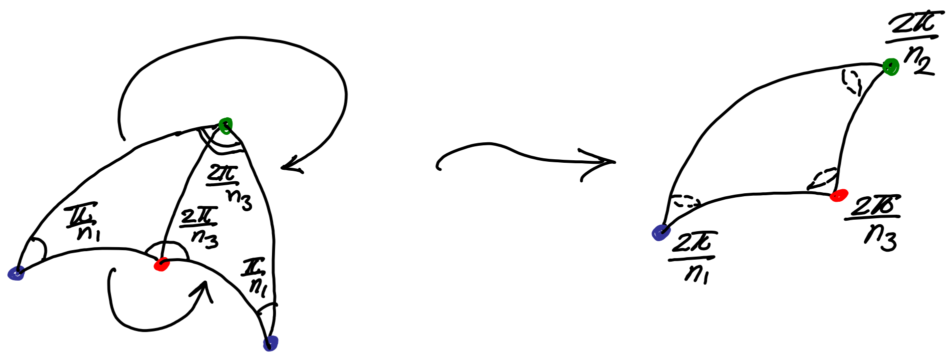

$R_1, R_2, R_3$ are reflections with respect to the sides.
Let $\Lambda = \langle R_1, R_2, R_3 \rangle$.

Since $n_i$'s are integers,
$\lambda_1 T^\circ \cap \lambda_2 T^\circ \neq \emptyset \Rightarrow \lambda_1 T^\circ = \lambda_2 T^\circ \Rightarrow \lambda_1 = \lambda_2$.
Also, $\bigcup_{\lambda \in \Lambda} \lambda \overline{T} = \mathbb{H}$, i.e., $T$ is a fundamental domain for $\Lambda$.

$\Lambda_0 = \{\text{even products of reflections}\} \underset{\text{index } 2}{\subset} \Lambda$.
Then $\Lambda_0 \subset PSL_2(\mathbb{R})$.

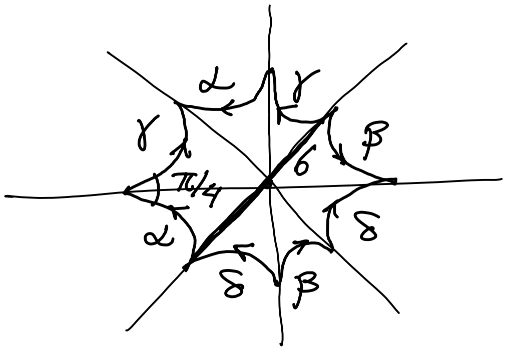
**Prop.** $\Lambda_0$ is discrete and cocompact.

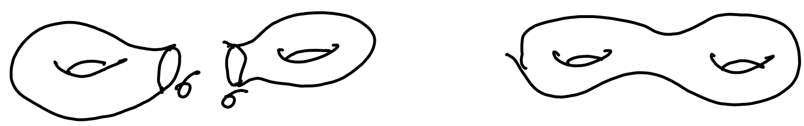
Consider the map $p: SL_2(\mathbb{R}) \to \mathbb{H}: g \mapsto g \cdot i$.
Note that $p$ is proper, and $p(g \cdot h) = g \cdot p(h)$.
If $F = T \cup R_1(T)$, then $\Lambda_0 \cdot F = \mathbb{H}$,
and $\Lambda_0 \cdot p^{-1}(F) = SL_2(\mathbb{R})$.
Hence, $\Lambda_0$ is cocompact.
If $K \subset SL_2(\mathbb{R})$ is compact, so is $p(K)$, and
$p(K)$ intersects only finitely many tiles $\lambda \cdot F$.
Then $K \cap \Lambda_0$ is finite, so $\Lambda_0$ is discrete. $\square$



3) <u>Genus-2 surface.</u>

Take regular 8-gon with angles $\frac{\pi}{4}$.



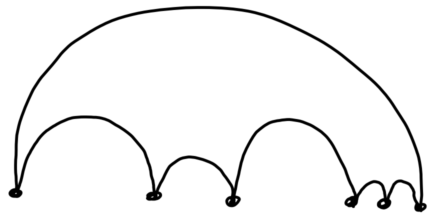
<u>Def</u> A discrete group $\Gamma < SL_2(\mathbb{R})$ is a <u>lattice</u> if $|F| < \infty$ for its fundamental domain.

<u>ex.</u> If $F_1$ & $F_2$ are fundamental domains, then $|F_1| = |F_2|$.

<u>Thm.</u> (Siegel) If $\Gamma$ is a lattice, then the Dirichlet fundamental domain has finitely many sides.

1) $D$ has finitely many vertices on $\mathbb{R}\cup\{\infty\}$.

If $|\partial D \cap (\mathbb{R}\cup\{\infty\})| \geq n$, by convexity, $D \supset$ $n$-gon with vertices on $\mathbb{R}\cup\{\infty\}$:

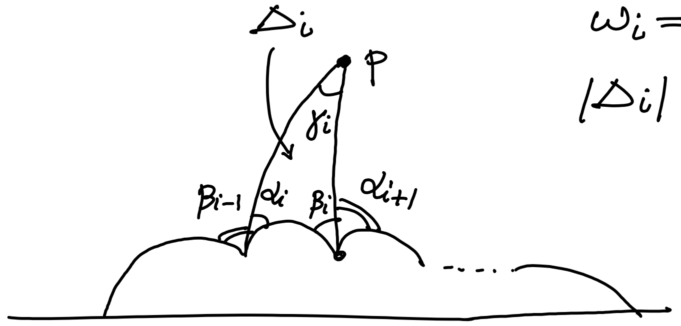


whose area is $\pi(n-2)$.

1) $\Rightarrow$ $\partial D$ has finitely many connected components

2) For all but finitely vertices, angles $\omega_i \geq \frac{3\pi}{4}$.

Consider a piece of $D$ corresponding to a connected component of $\partial D$:




$$\omega_i = \alpha_i + \beta_{i-1}$$

$$|\Delta_i| = \pi - (\alpha_i + \beta_i + \gamma_i)$$

$$\underbrace{\sum_{i=a}^{b} |\Delta_i|}_{\leq |D|} = \sum_{i=a}^{b} (\pi - \alpha_i - \beta_i - \gamma_i) = \pi - \alpha_a - \beta_b + \sum_{i=a+1}^{b-1} (\pi - \omega_i) - \underbrace{\sum_{i=a}^{b} \gamma_i}_{\leq 2\pi}$$

Hence, $\sum_i (\pi - \omega_i) < \infty \Rightarrow \omega_i \geq \frac{3\pi}{4}$ for all but finitely many $i$.

3) There are only finitely many vertices.

Take a vertex $v$ and $v_1, \dots v_m$ be the other vertices of $D$ in $\Gamma \cdot v$, i.e., $v_i = \gamma_i \cdot v$.


Let $\Gamma_v = \mathrm{Stab}_\Gamma(v)$.

Since $\Gamma$ is discrete, $\Gamma_v$ is finite.

All the copies of the tesselation $\gamma D$ adjucent to v are $\gamma \cdot \gamma_i^{-1} D$ with $\gamma \in \Gamma_v$.

Hence, $2\pi = |\Gamma_v| \cdot (\omega + \omega_1 + \dots + \omega_n)$

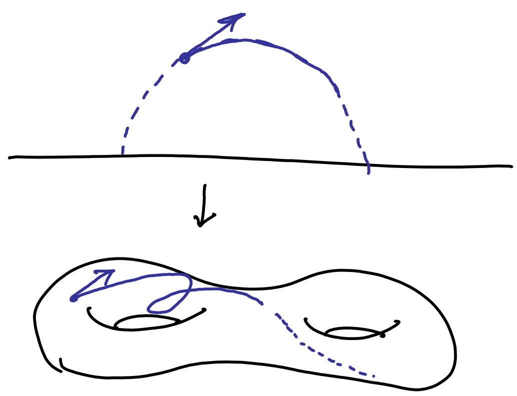
It follows from 2) that there are only finitely many vertices.

# Lecture 3: Geodesic flow.

$\Gamma < SL_2(\mathbb{R})$ – lattice subgroup,

$M = \Gamma \backslash \mathbb{H}$ – hyperbolic subgroup, $|M| < \infty$.

## Geodesic flow



$T^1\mathbb{H} \cong PSL_2(\mathbb{R})$

$T^1 M \cong \Gamma \backslash PSL_2(\mathbb{R})$

$g_t : T^1M \to T^1M$

$$g_t : \Gamma\backslash PSL_2(\mathbb{R}) \to \Gamma\backslash PSL_2(\mathbb{R})$$
$$x \longmapsto x \cdot \underbrace{\begin{pmatrix} e^{t/2} & 0 \\ 0 & e^{-t/2} \end{pmatrix}}_{a_t}$$

$$X = T^1M \simeq \Gamma\backslash PSL_2(\mathbb{R}).$$

Let $\mu = \frac{dx\,dy}{y^2}\,d\theta$ be the measure on $T^1M$, where $d\theta$ is the Lebesgue measure on $S^1$.

For $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{R})$, $g : (z, v) \mapsto \left(\frac{az+b}{cz+d}, \frac{v}{(cz+d)^2}\right)$.

It is easy to check that $\mu$ is left $SL_2(\mathbb{R})$-inv.,

i.e. defines left inv. measure on $PSL_2(\mathbb{R})$.
Because of invariance, $\mu$ also gives a measure on $X = \Gamma \backslash T^1 M$.
Since $|M| < \infty$, $\mu(X) < \infty$.
We normalise $\mu$ so that $\mu(X) = 1$.

In fact, $\mu$ is also right invariant.
(one can deduce this from uniqueness of inv. measure on $PSL_2(\mathbb{R})$).
In particular, $\mu$ is $g_t$-inv.

<u>Def</u> 1) $g_t$ is <u>ergodic</u> if for every measurable $A \subset X$ which is $g_t$-inv., we have $\mu(A) = 0$ or $1$.

2) $g_t$ is <u>mixing</u> if for every measurable $A, B \subset X$,

$$\mu(A \cap g_t^{-1} B) \to \mu(A)\mu(B) \text{ as } t \to \infty.$$

<u>Thm.</u> The geodesic flow is mixing.

For $g \in SL_2(\mathbb{R})$ and $\varphi \in L^2(X)$, define

$$\pi(g)\varphi(x) = \varphi(x \cdot g).$$

Note that $\pi(g_1 g_2) = \pi(g_1)\pi(g_2)$.

By invariance of $\mu$, $\|\pi(g)\varphi\|_2^2 = \int_X |\varphi(xg)|^2 \, d\mu(x) = \|\varphi\|_2^2$,

i.e., $\pi(g) : L^2(X) \to L^2(X)$ is unitary.

Since $\mu(A \cap g_t^{-1} B) = \langle \chi_A, \pi(a_t)\chi_B \rangle$,

mixing is equivalent to

$$\langle \varphi, \pi(a_t)\psi \rangle \xrightarrow[t\to\infty]{} \left(\int_X \varphi \, d\mu\right)\left(\int_X \bar{\psi} \, d\mu\right).$$

for all $\varphi, \psi \in L^2(X)$.

<u>Lem.</u> The map $SL_2(\mathbb{R}) \to L^2(X) : g \mapsto \pi(g)\varphi$ is continuous.

<u>Hint:</u> $C_c(X) \subset L^2(X)$ is dense.

<u>Proof of Thm.</u>

Without loss of generality, $\int_X \varphi \, d\mu = \int_X \psi \, d\mu = 0$.

Suppose then that for some $\varphi, \psi$ and $t_n \to \infty$,

$$\langle \varphi, \pi(a_{t_n})\psi \rangle \not\to 0.$$

$\underset{\sim\sim\sim}{\text{Weak convergence:}}$ $\psi_n \xrightarrow{weak} \psi$ if $\langle \varphi, \psi_n \rangle \to \langle \varphi, \psi \rangle$ for all $\varphi \in L^2(X)$.

$\underset{\sim\sim\sim}{\text{Banach-Alaoglu Thm}}$ Closed bounded subsets of $L^2(X)$ are compact in weak topology.

Since $\|\pi(a_{t_n})\psi\| = \|\psi\|$, passing to a subsequence, we may assume that $\pi(a_{t_n})\psi \to \tilde{\psi} \in L^2(X)$.

Claim $\tilde{\psi}$ is invariant under $U = \{u_s = \begin{pmatrix} 1 & s \\ 0 & 1 \end{pmatrix}\}$.

We use that
$$a_{t_n}^{-1} u_s a_{t_n} = \begin{pmatrix} e^{t_n/2} & 0 \\ 0 & e^{-t_n/2} \end{pmatrix}^{-1} \begin{pmatrix} 1 & s \\ 0 & 1 \end{pmatrix} \begin{pmatrix} e^{t_n/2} & 0 \\ 0 & e^{-t_n/2} \end{pmatrix}$$
$$= \begin{pmatrix} 1 & e^{-t_n} s \\ 0 & 1 \end{pmatrix} = u_{e^{-t_n}s} \to e.$$

$$\pi(u_s)\tilde{\psi} = \underset{n\to\infty}{w\text{-}\lim}\ \pi(u_s)\pi(a_{t_n})\psi = \underset{n\to\infty}{w\text{-}\lim}\ \pi(a_{t_n})\pi(u_{e^{-t_n}s})\psi$$

$$\|\pi(a_{t_n})\pi(u_{e^{-t_n}s})\psi - \pi(a_{t_n})\psi\| = \|\pi(u_{e^{-t_n}s})\psi - \psi\| \to 0.$$

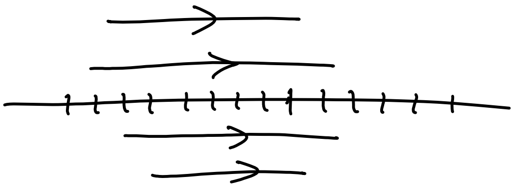
Hence, $\pi(u_s)\tilde{\psi} = \underset{n\to\infty}{w\text{-}\lim}\ \pi(a_{t_n})\psi = \tilde{\psi}$. $\square$

Consider the function $F(g) = \langle \pi(g)\tilde{\psi}, \tilde{\psi} \rangle$, $g \in SL_2(\mathbb{R})$.

Then $F(U \cdot g \cdot U) = F(g)$.

Note that $SL_2(\mathbb{R})/U \simeq \mathbb{R}^2\setminus\{0\} : gU \mapsto ge_1$,
so $F$ gives a $U$-inv. function on $\mathbb{R}^2\setminus\{0\}$.

$U \curvearrowright \mathbb{R}^2 : \begin{pmatrix} x \\ y \end{pmatrix} \to \begin{pmatrix} x+sy \\ y \end{pmatrix}$



Orbits $\begin{cases} \text{lines } y=c,\ c\neq 0 \\ \text{points } \{(x,0)\}. \end{cases}$

$F = \text{const}$ on $y=c$, $c\neq 0$, and by continuity, $F = \text{const}$ on $y=0$.

Then $\langle \pi(a_t)\tilde{\varphi}, \tilde{\varphi}\rangle = F(a_t\cdot e_1) = F(e^{t/2}\cdot e_1) = F(e) = \|\tilde{\varphi}\|^2$.

This gives equality in the Cauchy-Schwarz inequality, so $\pi(a_t)\tilde{\varphi} = \lambda\tilde{\varphi}$ and $\lambda = 1$.

Hence, $F$ is $B$-biinvariant, $B = \left\{\begin{pmatrix} a & x \\ 0 & a^{-1}\end{pmatrix} : a>0\right\}$.

Since $B\cdot e_2 = \{y>0\}$ and $B\cdot(-e_2) = \{y<0\}$, $F$ is constant on $\{y>0\}$ and $\{y<0\}$, and by continuity, $F$ is constant, i.e.,

$$\langle \pi(g)\tilde{\varphi}, \tilde{\varphi}\rangle = \|\tilde{\varphi}\|^2 \text{ for } g\in SL_2(\mathbb{R}).$$

As above, we deduce that $\pi(g)\tilde{\varphi} = \tilde{\varphi}$ for all $g$.

This means that for all $g$ and a.e. $x$, $\tilde{\psi}(gx) = \tilde{\psi}(x)$.
Then by Fubini Thm, for a.e. $x \in X$,
$\{g : \tilde{\psi}(gx) = \tilde{\psi}(x)\}$ has full measure.
Since $G$ acts transitively, $\tilde{\psi} = \text{const}$ in $L^2(X)$.
We have $\langle \varphi, \pi(a_{t_n})\psi \rangle \longrightarrow \langle \varphi, \tilde{\psi} \rangle \neq 0$,
but $\underset{\overset{\shortparallel}{0}}{\langle 1, \pi(a_{t_n})\psi \rangle} \longrightarrow \langle 1, \tilde{\psi} \rangle$.
This is a contradiction. $\lrcorner$

<u>Cor.</u> For a.e. $x \in X$, $\{g_t x\}_{t \geq 0}$ is dense in $X$.

## <u>Distribution of orbits.</u>

<u>Mean Ergodic Thm.</u> $\forall \varphi \in L^2(X)$:

$$\left\| \frac{1}{T} \int_0^T \varphi(g_t x)\,dt - \int_X \varphi\,d\mu \right\|_2 \xrightarrow[T \to \infty]{} 0.$$

$\ulcorner$ This follows from <u>mixing</u>. $\lrcorner$

**Pointwise Ergodic Thm.** $\forall \varphi \in L^2(X)$ $\forall$a.e. $x \in X$:

$$\frac{1}{T}\int_0^T \varphi(g_t \cdot x)\,dt \xrightarrow[T\to\infty]{} \int_X \varphi\, d\mu.$$

## Anosow property

Let $u_a = \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix}$ and $v_b = \begin{pmatrix} 1 & 0 \\ b & 1 \end{pmatrix}$.

Stable folliation: $W^s(x) = \{x \cdot u_a : a \in \mathbb{R}\}$.

Unstable folliation: $W^u(x) = \{x \cdot v_b : b \in \mathbb{R}\}$.

**Properties:**

1) (transversality) $T_x(g_{\mathbb{R}} x) \oplus T_x W^s(x) \oplus T_x W^u(x) = T_x M$.

2) $g_t W^s(x) = W^s(g_t x)$, $g_t W^u(x) = W^u(g_t x)$

$$\left[\; x u_a a_t = x \cdot a_t \cdot u_{e^{-t} a} \;\right]$$

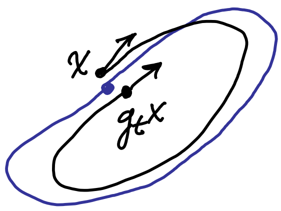
3) (contraction)
for $y \in W^s(x)$, $d(g_t x, g_t y) \leq \text{const} \cdot e^{-t}\, d(x,y)$;
for $y \in W^u(x)$, $d(g_{-t} x, g_{-t} y) \leq \text{const} \cdot e^{-t}\, d(x,y)$.

Let $y = x \cdot u_a$. Then

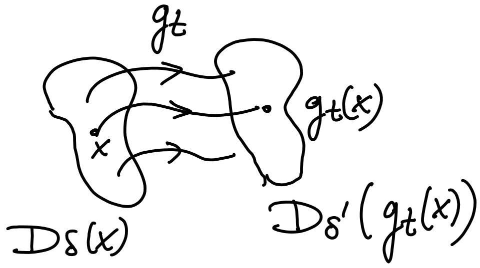
$$d(g_t(x), g_t(y)) = d(xa_t, xu_a \cdot a_t) = d(xa_t, xa_t \cdot u_{ae^{-t}}) \leq d(e, u_{ae^{-t}}) \leq \text{const} \cdot ae^{-t}.$$

## Anosov Closing Lemma.

Suppose that for some $x \in X$ and $t \geq 0$, $d(x, g_t(x)) < \varepsilon$.
Then $\exists$ periodic $x_0 \in X$: $d(x, x_0) \leq \text{const} \cdot \varepsilon$.



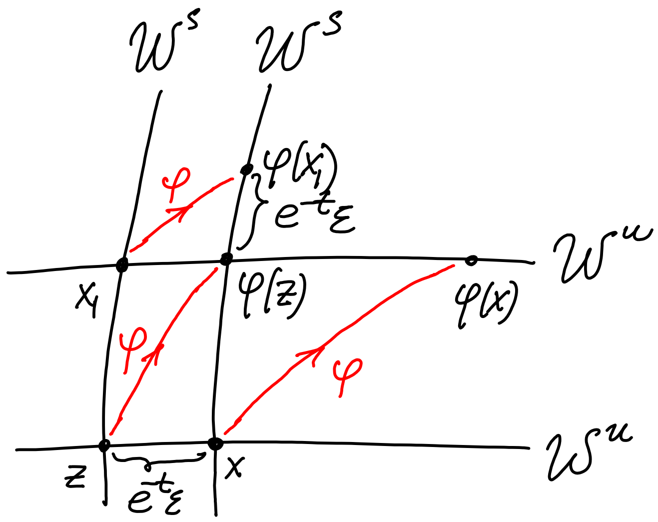

Let $D_\delta(x) = \{x \cdot u_a v_b : |a|, |b| < \delta\}$.
$D_\delta(x)$ is 2-dim. manifold transveral to the flow; $g_t(D_\delta(x)) \subset D_{\delta'}(g_t(x))$.




$\exists t_0 : |t_0| \leq \text{const}\, \varepsilon$ : $g_{t+t_0}(x) \in D_\delta(x)$.

Consider the map:

$$\varphi : D_\delta(x) \to D_{\delta'}(x)$$
$$y \longmapsto g_{t+t_0}(y)$$




By transversality, $\exists\, z \in W^u(x) : \ \varphi(z) \in W^s(x)$.

Then $$d(z,x) = d(g_{-t-t_0}(\varphi(z)), g_{-t-t_0}(\varphi(x))) < \text{const} \cdot e^{-t} \cdot \varepsilon.$$

Take $x_1 \in W^s(z) \cap W^u(\varphi(z))$.

Then $$d(\varphi(x_1), \varphi(z)) \leq d(g_{t+t_0}(x_1), g_{t+t_0}(z)) < \text{const} \cdot e^{-t} \cdot \varepsilon$$

Hence, $d(x_1, \varphi(x_1)) < \text{const} \cdot e^{-t} \cdot \varepsilon$.

Continuing this process, we construct $x_\infty$ such that $\varphi(x_\infty) = x_\infty$. This gives a periodic orbit. $\square$

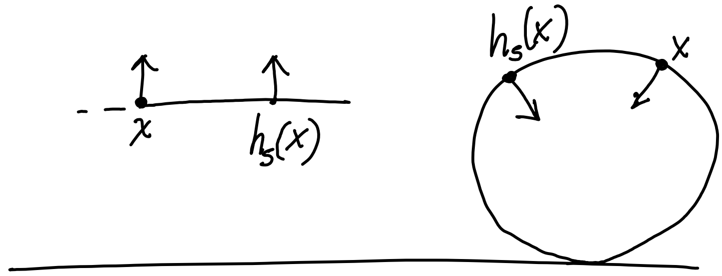
**Cor.** Periodic orbits of geodesic flow are dense.

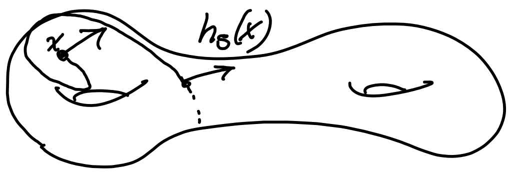
# Lecture 4: Horocycle flows.

$\Gamma$ – lattice subgroup of $SL_2(\mathbb{R})$

$X = \Gamma \backslash SL_2(\mathbb{R})$

Horocycle flows: $h_s : X \to X : x \mapsto x \cdot \begin{pmatrix} 1 & s \\ 0 & 1 \end{pmatrix}$.




Thm The horocycle flows $h_s$ is mixing.

Lem. (Cartan decomposition)

$$SL_2(\mathbb{R}) = SO(2) \cdot \left\{ a_t = \begin{pmatrix} e^{t/2} & 0 \\ 0 & e^{-t/2} \end{pmatrix} : t \geq 0 \right\} \cdot SO(2).$$

<u>Proof of Thm.</u> Suppose for some $\varphi, \psi \in L^2(X)$ and $s_n \to \infty$, $\langle \varphi, \pi(h_{s_n})\psi \rangle \not\to \int_X \varphi \, d\mu \cdot \int_X \overline{\psi} \, d\mu$.

By the Cartan decomposition,

$h_{s_n} = k_n a_{t_n} \ell_n$ with $k_n, \ell_n \in SO(2)$ and $t_n \to \infty$.

Then $\langle \varphi, \pi(h_{s_n})\psi \rangle = \langle \pi(k_n)^{-1} \cdot \varphi, \pi(a_{t_n}) \pi(\ell_n)\psi \rangle$.

Passing to a subsequence, $k_n \to k$ and $\ell_n \to \ell$,

so that

$$|\langle \pi(k_n)^{-1}\varphi, \pi(a_{t_n})\pi(\ell_n)\psi \rangle - \langle \pi(k)^{-1}\varphi, \pi(a_{t_n})\pi(\ell)\psi \rangle|$$

$$\leq |\langle \pi(k_n)^{-1}\varphi - \pi(k)^{-1}\varphi, \pi(a_{t_n})\pi(\ell_n)\psi \rangle|$$

$$+ |\langle \pi(k)^{-1}\varphi, \pi(a_{t_n})\pi(\ell_n)\psi - \pi(a_{t_n})\pi(\ell)\psi \rangle|$$

$$\leq \|\pi(k_n)^{-1}\varphi - \pi(k)^{-1}\varphi\| \cdot \|\psi\| + \|\varphi\| \cdot \|\pi(\ell_n)\psi - \pi(\ell)\|$$

$$\to 0.$$

Then $\langle \pi(k)^{-1}\varphi, \pi(a_{t_n})\pi(\ell)\psi \rangle \not\to \int_X \varphi \, d\mu \cdot \int_X \psi \, d\mu$,

which contradicts mixing of $g_t$. $\square$

**Thm** (unique ergodicity)
Assume that $X$ is compact.
Then $\forall x \in X$ $\forall f \in C(X)$:

$$\frac{1}{T}\int_0^T f(h_s(x))\,ds \longrightarrow \int_X f\,d\mu.$$

Let $B = \begin{pmatrix} * & 0 \\ * & * \end{pmatrix} \subset Sl_2(\mathbb{R})$ and $B_\delta = \delta$-nbhd of $e$ in $B$.
Then $G_\delta = h_{[-\delta,\delta]} B_\delta$ is a nbhd of $e$ in $G$.
Since $X$ is compact, $\exists \delta_0 > 0$: $G_{\delta_0} \to y \cdot G_{\delta_0}$ is injective for all $y \in X$. (check this).

The measure $\mu|_{yG_\delta}$ is given by $ds \cdot db$ where $db$ is right-invariant measure on $B$.

Let $Q = h_{[0,\delta_0]} \cdot B_\delta$ and $Q_t = a_t Q a_t^{-1} = h_{[0,e^t\delta_0]}\underbrace{(a_t B_\delta a_t^{-1})}_{B_t}$.
Note that $a_t \begin{pmatrix} u & 0 \\ w & v \end{pmatrix} a_t^{-1} = \begin{pmatrix} u & 0 \\ e^{-t}w & v \end{pmatrix}$, so
$B_t$ is $\delta$-small and $x \cdot Q_t$ is "thickening" of the orbit $x \cdot h_{[0,e^t\delta_0]}$.
By injectivity, $|Q_t| = |Qa_t^{-1}| = |Q|$.

By uniform continuity,

$\forall \varepsilon > 0: \exists \delta > 0: \forall y \in X: \forall b \in B_t: \ |f(y \cdot b) - f(y)| < \varepsilon.$

This suggests that $\displaystyle \frac{1}{|yQ_t|} \int_{yQ_t} f \, d\mu \approx \frac{1}{e^t \delta_0} \int_0^{e^t \delta_0} f(y \cdot a_s) ds.$

Indeed, $$\frac{1}{|yQ_t|} \cdot \int_{yQ_t} f \, d\mu = \frac{1}{|yQ_t|} \cdot \int_0^{e^t\delta_0} \int_{B_t} f(y u_s b) \, db \, ds$$

(*) $$= \frac{|B_t|}{e^t \delta_0 \cdot |B_t|} \cdot \int_0^{e^t \delta_0} \left( f(y u_s) + O(\varepsilon) \right) ds$$

$$= \frac{1}{e^t \delta_0} \int_0^{e^t \delta_0} f(y u_s) ds + O(\varepsilon).$$

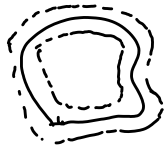
We claim that it follows from mixing (for geodesic flows)

that $\displaystyle \frac{1}{|yQ_t|} \cdot \int_{yQ_t} f \, d\mu = \left\langle f, \frac{\chi_{yQ_t}}{|yQ_t|} \right\rangle \to \int_X f \, d\mu.$

Without loss of generality, $f \geq 0$.

Pick compact $Q^-$ and open $Q^+$ such that

$Q^- \subset Q \subset Q^+$



$|Q^+ - Q^-| < \varepsilon.$

$\exists \rho>0$: $G_\rho \cdot Q^- \subset Q$ and $G_\rho \cdot Q \subset Q^+$.

By compactness, $X = z_1 G_\rho \cup \dots \cup z_\ell G_\rho$.

Assuming that $x a_t \in z_i G_\rho$, we obtain:

$$\left\langle f, \frac{\chi_{xQ_t}}{|Q_t|} \right\rangle = \left\langle f, \frac{\chi_{x a_t Q a_t^{-1}}}{|Q_t|} \right\rangle \leq \left\langle f, \frac{\chi_{z_i G_\rho Q a_t^{-1}}}{|Q_t|} \right\rangle$$

$$\leq \left\langle f, \frac{\pi(a_t)\chi_{z_i Q^+}}{|Q_t|} \right\rangle \xrightarrow[t\to\infty]{} \left(\int_X f\right) \cdot \frac{|Q^+|}{|Q_t|}.$$

This proves that

$$\varlimsup_{t\to\infty} \left\langle f, \frac{\chi_{xQ_t}}{|Q_t|} \right\rangle \leq \int_X f \cdot (1+\varepsilon)$$

for all $\varepsilon>0$.

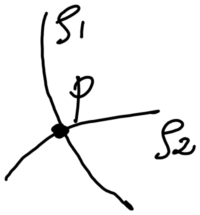
The lower bound is proved similarly.

Hence, the Thm follows from $(*)$. $\square$

<u>Cor.</u> For all $x\in X$, the orbit $\{h_s(x)\}_{s>0}$ is dense.

# Lecture 5.

## Laplace operator and its spectral decomposition.




Let $f \in C^2(\mathbb{H})$.

For $p \in \mathbb{H}$, take orthogonal geodesics $\rho_1, \rho_2$ with $\rho_1(0) = \rho_2(0) = p$.

The <u>Laplace operator</u>:

$$\Delta f(p) = \frac{d^2 f(\rho_1(t))}{dt^2}\Big|_{t=0} + \frac{d^2 f(\rho_2(t))}{dt^2}\Big|_{t=0}.$$

Explicitly, the Laplace operator is given by

$$\Delta = -y^2\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right).$$

We note that $\Delta$ is the unique (up to scalar multiple) 2nd order differential operator that <u>commutes</u> with $SL_2(\mathbb{R})$-action.

In particular, for a hyperbolic surface $M = \Gamma \backslash \mathbb{H}$, we have $\Delta : C^\infty(M) \to C^\infty(M)$.

Let $M = \Gamma \backslash \mathbb{H}$ be a compact hyperbolic surface.

**Thm** (Spectral decomposition)
$L^2(M)$ has an orthonormal basis $\varphi_0, \dots \varphi_n, \dots$ of $C^\infty$-functions with eigenvalues $0 = \lambda_0 < \lambda_1 \leq \dots \leq \lambda_n \leq \dots$ with $\lambda_n \to \infty$.

**Prop.** 1) $\Delta$ is symmetric: $\langle \Delta f_1, f_2 \rangle = \langle f_1, \Delta f_2 \rangle$, $f_1, f_2 \in C^\infty(M)$.
2) $\Delta$ is positive: $\langle \Delta f, f \rangle \geq 0$; "$=$" $\Leftrightarrow$ $f = \text{const}$.
$f \in C^\infty(M)$

Consider the differential form:

$$\omega = f_1 \left( \frac{\partial \overline{f_2}}{\partial x} dy - \frac{\partial \overline{f_2}}{\partial y} dx \right) - \overline{f_2} \left( \frac{\partial f_1}{\partial x} dy - \frac{\partial f_1}{\partial y} dx \right).$$

Note that:

1) $\omega$ is $\Gamma$-invariant (check using the Cauchy-Riemann equations for $z \mapsto \gamma z$)

2) $d\omega = -(f_1 \Delta^e \overline{f_2})\, dx \wedge dy + (\Delta^e f_1 \cdot \overline{f_2})\, dx \wedge dy$,
where $\Delta^e = -\frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2}$ is the Euclidean Laplacian.

By Stoke's Thm, $\int_M d\omega = 0$, and

$$\underbrace{\int_M \Delta^e f_1 \cdot \overline{f_2}\, dx dy}_{\langle \Delta f_1, f_2 \rangle} = \underbrace{\int_M f_1 \cdot \Delta^e \overline{f_2}\, dx dy}_{\langle f_1, \Delta f_2 \rangle}.$$

To prove (2), we apply Stoke's Thm to:

$$\omega = \overline{f}\left(\frac{\partial f}{\partial x} dy - \frac{\partial f}{\partial y} dx\right).$$

This form is $\Gamma$-inv., and

$$d\omega = \left(\left|\frac{\partial f}{\partial x}\right|^2 + \left|\frac{\partial f}{\partial y}\right|^2 - \overline{f}\, \Delta^e f\right) dx \wedge dy.$$

Then $\underbrace{\int_M \Delta^e f \cdot \overline{f}\, dx dy}_{\langle \Delta f, f \rangle} = \int_M \left(\left|\frac{\partial f}{\partial x}\right|^2 + \left|\frac{\partial f}{\partial y}\right|^2\right) dx dy \geq 0.$

Hence, $\langle \Delta f, f \rangle \geq 0$ with "$=$" $\Leftrightarrow$ $f = \text{const}$. $\square$

The Spectral Thm is proved by studing:

<u>Heat equation:</u> For $f \in C^\infty(M)$, find $u \in C^\infty(M \times \mathbb{R}^+)$,

$$\begin{cases} \Delta u + \frac{\partial u}{\partial t} = 0 \\ u|_{t=0} = f. \end{cases}$$

$u(z,t)$ = temperature at time at $z$, when the initial temperature is $f$.

**Def.** A $C^\infty$-function $p : M \times M \times (0,\infty) \to \mathbb{R}$ is called a heat kernel if:

1) $\Delta p_t(\cdot, w) + \frac{\partial p_t}{\partial t} = 0$,
2) $p_t(z,w) = p_t(w,z)$,
3) $\lim\limits_{t \to 0^+} \int_M p_t(z,w) f(w)\, dm(w) = f(z)$.

---

Note that $u(z,t) = \int_M p_t(z,w) f(w)\, dm(w)$ gives a solution for the heat equation.

**Prop.** A solution of the heat equation is unique.

If $u_1, u_2$ are solutions, then $v = u_1 - u_2$ is a solution with $f = 0$. We note that

$$\frac{d}{dt}\left(\int_M v^2\right) = 2\langle v, \frac{\partial v}{\partial t}\rangle = -2\langle v, \Delta v\rangle \leq 0.$$

Since $v|_{t=0} = 0$, $\int_M v^2 \leq 0$ for $t \geq 0 \Rightarrow v = 0$.

Cor. 1) The heat kernel is unique.

2) $\int_M p_t(z,w)\,d\mu(w) = 1.$

Proof of Spectral decomposition:
(assuming existance of heat kernel)

We consider a family of operators:

$$\mathcal{P}_t : L^2(M) \longrightarrow L^2(M)$$
$$f \longmapsto \int_M p_t(z,w) f(w)\,dm(w)$$

Properties: 0) $\mathcal{P}_t f$ is a solution of heat equation

1) $\mathcal{P}_{t_1} \mathcal{P}_{t_2} = \mathcal{P}_{t_1+t_2}$
(this follows from uniqueness of solutions of heat equations)

2) $\langle \mathcal{P}_t f, f\rangle = \langle \mathcal{P}_{t/2} f, \mathcal{P}_{t/2} f\rangle \geq 0.$

By the Hilbert-Schmidt Thm, $\mathcal{P}_t$ is diagonalisable, i.e., $L^2(M)$ has an orthonormal basis of eigenfunctions of $\mathcal{P}_t$.
Let $\varphi_0, \varphi_1, \dots \varphi_n, \dots$ be the eigenbasis for $\mathcal{P}_1$ with eigenvalues $\eta_0 \geq \dots \geq \eta_n \geq \dots \geq 0$, $\eta_n \to 0$.
We shall shows that $\{\varphi_i\}$ is eigenbasis for $\Delta$.

If $P_{1/k}\varphi = \eta\varphi$, then $P_1\varphi = \eta^k\varphi$.

This implies that the eigenspaces of $P_{1/k}$ and $P_1$ coinside, and $P_{\ell/k}\varphi_i = \eta_i^{\frac{\ell}{k}}\varphi_i$.

Then by continuity, $P_t\varphi_i = \eta_i^t\varphi_i$ for all $t>0$.

By the properties of heat kernel,

$$P_t\varphi_i = \eta_i^t\varphi_i \xrightarrow[t\to\infty]{} \varphi_i \Rightarrow \eta_i > 0.$$

In particular, $\varphi_i = \eta_i^{-1}P_t\varphi_i \in C^\infty(M)$.

We note that $\varphi = \text{const}$ is an eigenfunction of $P_1$ with eigenvalue 1. Let $\varphi$ be $\neq$ const eigenfunction of $P_1$ with eigenvalue $\eta$. Then

$$\frac{d}{dt}\langle P_t\varphi, P_t\varphi\rangle = 2\left\langle \frac{d}{dt}P_t\varphi, P_t\varphi\right\rangle = -2\langle \Delta P_t\varphi, P_t\varphi\rangle$$
$$= -2\eta^{2t}\langle \Delta\varphi, \varphi\rangle < 0.$$

Hence, $\|P_t\varphi\| = \eta^t\|\varphi\|$ decays, so that $\eta<1$.

Finally, we claim that $\Delta\varphi_i = \lambda_i\varphi_i$ with $\lambda_i = -\log\eta_i$.

Indeed, $0 = \Delta P_t\varphi_i + \frac{\partial}{\partial t}P_t\varphi_i = \Delta\left(e^{-\lambda_i t}\varphi_i\right) + \frac{\partial}{\partial t}\left(e^{-\lambda_i t}\varphi_i\right)$

$$= e^{-\lambda_i t}\left(\Delta\varphi_i - \lambda_i\varphi_i\right).$$

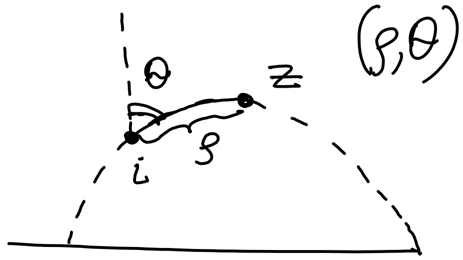
## Heat kernel on $\mathbb{H}$.

We are looking for $p_t : \mathbb{H}\times\mathbb{H} \to \mathbb{R}$ satisfying the properties of heat kernel. We expect:

1) $p_t(z,w)$ depends only on $d(z,w)$, namely,
$$p_t(z,w) = P_t(d(z,w)) \text{ for } P_t : [0,\infty) \to \mathbb{R}.$$

2) $P_t(\rho) \to 0$ rapidly as $\rho \to \infty$.

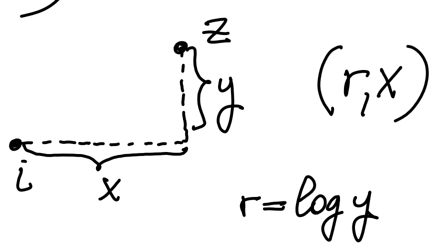
Then it is convenient to use polar coordinates:




$$\Delta = -\frac{\partial^2}{\partial \rho^2} - \coth(\rho)\cdot\frac{\partial}{\partial \rho} + (\dots)$$

Then $$-\frac{\partial^2 P_t}{\partial \rho^2} - \coth(\rho)\frac{\partial P_t}{\partial \rho} + \frac{\partial P_t}{\partial t} = 0.$$

However, $\Delta$ is simpler in horospherical coordinates:




$$\Delta = -\frac{\partial^2}{\partial r^2} + \frac{\partial}{\partial r} + (\dots)$$

For a rapidly decaying $f: \mathbb{H} \to \mathbb{R}$, we consider "horospherical transform":

$$(Hf)(z) = \int_{\mathbb{R}} f(z+s)\,ds.$$

Since $Hf$ depends only on $r$,

$$\begin{tikzcd}
C^\infty(\mathbb{H}) \arrow[r, "H"] \arrow[d, "\Delta"] & C^\infty(\mathbb{H}) \arrow[r, "\times e^{-r/2}"] \arrow[d, "-\frac{\partial^2}{\partial r^2}+\frac{\partial}{\partial r}"] & C^\infty(\mathbb{H}) \arrow[d, "-\frac{\partial^2}{\partial r^2}+\frac{1}{4}"] \\
C^\infty(\mathbb{H}) \arrow[r, "H"] & C^\infty(\mathbb{H}) \arrow[r, "\times e^{-r/2}"] & C^\infty(\mathbb{H})
\end{tikzcd}$$

First, we compute:

$$Q_t(r) = e^{-r/2}\, H[P_t(i,z)],$$

which satisfies the equation:

$$-\frac{\partial^2 Q_t}{\partial r^2} + \frac{1}{4}\cdot Q_t + \frac{\partial Q_t}{\partial t} = 0$$

Since it is similar to the heat equation on $\mathbb{R}$, we look for solutions of the form

$$Q_t(r) = \alpha(t)\cdot e^{-\beta r^2/t},$$

and find $\boxed{Q_t(r) = \text{const}\cdot t^{-1/2}\cdot e^{-t/4}\cdot e^{-r^2/4t}.}$ $(*)$

On the othee hand,

$$Q_t(r) = e^{-r/2} \cdot \int_{\mathbb{R}} P_t(d(i, s+ie^r))\,ds$$

Using that $\cosh d(u,v) = 1 + \frac{|u-v|^2}{2\,Im(u)\cdot Im(v)}$,

$$\cosh d(i, s+ie^r) = 1 + \frac{s^2+(e^r-1)^2}{2e^r} = \cosh r + \tfrac{1}{2}s^2 \cdot e^{-r}.$$

We make change of variables,

$$\rho = d(i, s+ie^r) = \cosh^{-1}\left(\cosh r + \tfrac{1}{2}s^2 \cdot e^{-r}\right).$$

Then $s = \sqrt{2e^r(\cosh\rho - \cosh r)}$ $\quad \sinh\rho\, d\rho = s \cdot e^{-r} dr.$

,

$$\boxed{Q_t(r) = \sqrt{2} \cdot \int_r^\infty \frac{P_t(\rho)\sinh\rho\, d\rho}{\sqrt{\cosh\rho - \cosh r}}} \qquad (**)$$

Def. Abel transform: for rapidly decaying $p : [1,\infty) \to \mathbb{R}$,

$$a[p](x) = \int_x^\infty \frac{p(y)\,dy}{\sqrt{y-x}} \underset{\xi=\sqrt{y-x}}{\overset{\uparrow}{=}} 2 \cdot \int_0^\infty p(x+\xi^2)\,d\xi.$$

**Lem.** $\displaystyle a^{-1}[q](x) = -\frac{1}{\pi}\int_x^\infty \frac{q'(y)\,dy}{\sqrt{y-x}}$

$$p(x) = -\int_0^\infty \left(p(x+\xi^2)\right)'_\xi d\xi = -2\int_0^\infty p'(x+\xi^2)\underbrace{\xi\, d\xi}_{\substack{\text{density in}\\ \text{polar coordinates}\\ \text{in } \mathbb{R}^2}}$$

$$= -\frac{4}{\pi}\int_0^\infty\int_0^\infty p'(x+u^2+v^2)\,du\,dv$$

$$= -\frac{2}{\pi}\int_0^\infty a[p'](x+v^2)\,dv = -\frac{2}{\pi}\int_0^\infty a[p]'(x+v^2)\,dv$$

$$\underset{\substack{\uparrow\\ y=x+v^2}}{=} -\frac{1}{\pi}\int_x^\infty \frac{a[p]'(y)}{\sqrt{y-x}}\,dx.$$

By $(*)$ and $(**)$,

$$P_t(\rho) = \text{const}\cdot\int_\rho^\infty \frac{Q_t'(r)}{\sqrt{\cosh r - \cosh\rho}}\,dr = \text{const}\cdot t^{-3/2}\cdot e^{-t/4}\int_\rho^\infty \frac{r\cdot e^{-r^2/4t}}{\sqrt{\cosh r-\cosh\rho}}\,dr,$$

and $p_t(z,w) = P_t(d(z,w))$.

We choose the constant so that

$$\int_{\mathbb{H}} p_t(z,w)\,dm(z) = 1. \quad (***)$$

**Lem.** $p_t(z,w) \leq \text{const} \cdot t^{-1} \cdot e^{-d(z,w)^2/8t}$. $(****)$

**Thm.** $$p_t(z,w) = \frac{\sqrt{2}}{(4\pi t)^{3/2}} \cdot e^{-t/4} \cdot \int_{d(z,w)}^{\infty} \frac{r \cdot e^{-r^2/4t}\, dr}{\sqrt{\cosh r - \cosh d(z,w)}}$$

is the heat kernel on $\mathbb{H}$.

By construction, $p_t(\cdot, w)$ satisfies the heat equation, and clearly, $p_t(z,w) = p_t(w,z)$.

The property: $\int_{\mathbb{H}} p_t(z,w) f(w)\, dm(w) \xrightarrow[t \to 0]{} f(z)$

is deduced from $(***)$ and $(****)$.

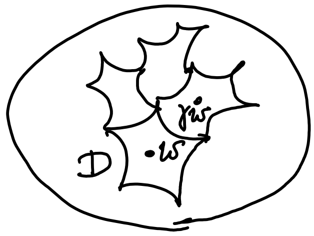
## Heat kernel on hyperbolic surfaces.

Let $M = \Gamma \backslash \mathbb{H}$ be a compact hyperbolic surface.

We set $\overline{p_t}(z,w) = \sum_{\gamma \in \Gamma} p_t(z, \gamma w)$.

**Lem.** $\#\{\gamma \in \Gamma : d(z, \gamma w) < T\} \leq C(z,w) \cdot e^T$.

$\ulcorner$




$B_{T+\mathrm{diam}(D)}(w)$

Since $d(z,\gamma w) \geq d(w,\gamma w) - d(z,w)$, we may assume that $z = w$. Let $D$ be the compact Dirichlet fundamental domain at $w$. Then

$$\#\{\gamma \in \Gamma : d(w,\gamma w) < T\} \leq \text{const} \cdot \Big| \bigcup_{\gamma \in \Gamma :\, d(w,\gamma w) < T} \gamma D \Big|$$
$$\leq \text{const} \, \big| B_{T+\mathrm{diam}(D)}(w) \big|$$
$$\leq \text{const} \cdot e^{T}. \quad \lrcorner$$

By Lemma and $(****)$,

$$\overline{p_t}(z,w) \leq \sum_{n=0}^{\infty} \#\{\gamma \in \Gamma : n \leq d(z,\gamma w) < n+1\} \cdot \text{const} \cdot t^{-1} \cdot e^{-n^2/8t}$$
$$\leq \text{const} \cdot \sum_{n=0}^{\infty} t^{-1} \cdot e^{n} \cdot e^{-n^2/8t} < \infty.$$

Similarly, the sum of derivatives of $p_t$ also converges. Hence, $\overline{p_t}$ satisfies the heat equation.

For $\gamma_1, \gamma_2 \in \Gamma$,

$$\overline{p_t}(\gamma_1 z, \gamma_2 w) = \sum_{\gamma \in \Gamma} p_t(\gamma_1 z, \gamma\gamma_2 w) = \sum_{\gamma \in \Gamma} p_t(z, \gamma_1^{-1}\gamma\gamma_2 w)$$
$$= \sum_{\gamma \in \Gamma} p_t(z, \gamma w) = \overline{p_t}(z,w).$$

Hence, $\overline{p_t}$ defines a function on $M \times M$.

Finally, for $\Gamma$-inv. $f : \mathbb{H} \to \mathbb{R}$,

$$\int_M \overline{p_t}(z,w) f(w)\, d\mu(w) = \int_D \sum_{\gamma \in \Gamma} p_t(z, \gamma w) f(w)\, dm(w)$$

$$= \int_{\mathbb{H}} p_t(z,w) f(w)\, dm(w) \xrightarrow[t \to 0]{} f(z)$$

# Lecture 6

## Trace formula and Weyl law.

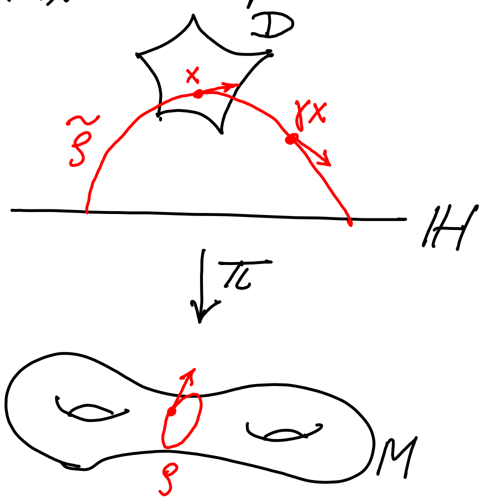
$M = \Gamma \backslash \mathbb{H}$ – compact hyperbolic surface.

$\mathcal{G}(M) = \{$ periodic geodesics in $T^1 M \}$

$$\boxed{\mathrm{Spec}(\Delta) \overset{?}{\longleftrightarrow} \mathcal{G}(M)}$$

**Prop.** $\#\{\rho \in \mathcal{G}(M) : \underbrace{|\rho|}_{\text{length}} < T\} \le \mathrm{const} \cdot e^T.$

Fix a compact fundamental domain $D \subset \mathbb{H}$ for $M$.




Given $\rho \in \mathcal{G}(M)$ with $|\rho| < T$, we take a geodesic line $\tilde{\rho} \subset \mathbb{H}$ which is a "lift" of $\rho$, (namely, $\pi(\tilde{\rho}) = \rho$ where $\pi : \mathbb{H} \to M$ is the factor map) and $\tilde{\rho} \cap D \neq \emptyset$.

Then $\exists x \in \tilde{\rho} \cap D$, $\gamma \in \Gamma$ : as in the picture with $d(x, \gamma x) < T$.

Then $\gamma(\tilde{\rho}) = \tilde{\rho}$. $\quad (*)$

We note that given $\gamma$, $(*)$ determines $\tilde{\rho}$ uniquely.
Indeed, $(*) \Rightarrow \gamma$ fixes the end-points of $\tilde{\rho}$.
This gives a quadratic equation $\gamma \cdot z = z$ which has at most 2 solutions.
Hence, the end-points (and $\tilde{\rho}$) are unique.

Hence, the above correspondence $\rho \longmapsto \gamma$ is 1-to-1, and

$$\#\{\rho \in \mathcal{G}(M) : |\rho| < T\} \leq \#\{\gamma \in \Gamma : d(x, \gamma x) < T \text{ for some } x \in \mathcal{D}\}$$

$$\leq \#\{\gamma \in \Gamma : d(z, \gamma z) < T + \operatorname{diam}(\mathcal{D})\}$$

$$\leq \text{const} \cdot e^T,$$

for a fixed $z \in \mathcal{D}$. $\lrcorner$

Def 1) $g \in SL_2(\mathbb{R})$ is *hyperbolic* if it can be conjugated to a diagonal matrix.

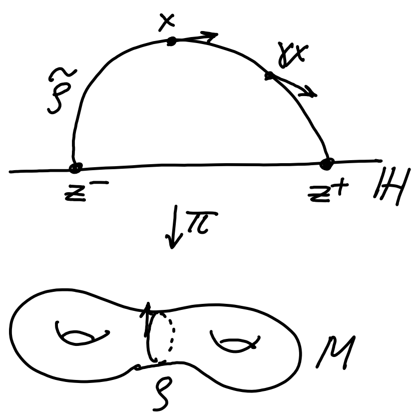
2) $\gamma \in \Gamma$ is *primitive* if $\gamma \neq \delta^k$ for some $k \geq 2$, $\delta \in \Gamma$.

We note every hyperbolic $g$ has 2 fixed points in $\mathbb{R} \cup \{\infty\}$ and fixes the unique geodesic

**Thm.** There is a 1-to-1 correspondence:

$$\mathcal{G}(M) \longleftrightarrow \left\{ \begin{array}{l} \text{congugacy classes in } \Gamma \\ \text{of primitive hyperbolic elements} \end{array} \right\}$$

$\ulcorner$




As in the previous argument, given $\rho \in \mathcal{G}(M)$, we take a geodesic line $\tilde{\rho} \subset \mathbb{H}$ such that $\pi(\tilde{\rho}) = \rho$, and $\gamma \in \Gamma \setminus \{e\}$ such that $\gamma \cdot \tilde{\rho} = \tilde{\rho}$.

Let $A_{\tilde{\rho}} = \mathrm{Stab}_{PSL_2(\mathbb{R})}(\tilde{\rho})$. We note that $A_{\{x=0\}} = \begin{pmatrix} * & 0 \\ 0 & * \end{pmatrix}$ and $A_{\tilde{\rho}}$ is conjugate to $\begin{pmatrix} * & 0 \\ 0 & * \end{pmatrix}$.
In particular, $A_{\tilde{\rho}} \simeq \mathbb{R}$. Since $A_{\tilde{\rho}} \cap \Gamma$ is discrete, it must be cyclic and $A_{\tilde{\rho}} \cap \Gamma = \langle \gamma_0 \rangle$ for $\gamma_0 \in \Gamma \setminus \{e\}$.
We claim that $\gamma_0$ is primitive. Indeed, if $\gamma_0 = \delta^k$ for some $\delta \in \Gamma$ and $k \geq 2$, then $\delta$ is also hyperbolic and $\delta \cdot \tilde{\rho} = \tilde{\rho}$, i.e. $\delta \in A_{\tilde{\rho}} \cap \Gamma$, but this is impossible because $\gamma_0$ is a generator $A_{\tilde{\rho}} \cap \Gamma \simeq \mathbb{Z}$.
Now we constructed a map $\rho \longmapsto \gamma_0$.

Conversely, given a primitive hyperbolic $\gamma_0 \in \Gamma$, it has 2 fixed points $z^-, z^+ \in \mathbb{R} \cup \{\infty\}$ (attracting & repelling), which determines a unique directed geodesic line $\tilde{\rho}$ and a unique closed geodesic in $T^1(M)$.

The correspondence $\tilde{\rho} \leftrightarrow \gamma_0$ is 1-to-1, but it might happen that

$$\pi(\tilde{\rho}_1) = \pi(\tilde{\rho}_2) \Leftrightarrow \exists \gamma \in \Gamma : \gamma \cdot \tilde{\rho}_1 = \tilde{\rho}_2 .$$

Then $A_{\tilde{\rho}_2} = \gamma A_{\tilde{\rho}_1} \gamma^{-1}$ and $\underset{\langle \gamma_2 \rangle}{\underset{\parallel}{A_{\tilde{\rho}_2}}} \cap \Gamma = \gamma \cdot \underbrace{(A_{\tilde{\rho}_1} \cap F)}_{\langle \gamma_1 \rangle} \cdot \gamma^{-1}$.

Hence, $\gamma_2^{\pm 1} = \gamma \cdot \gamma_1 \cdot \gamma^{-1}$.

In fact, $\gamma_2 = \gamma \cdot \gamma_1 \cdot \gamma^{-1}$ if we take directions of $\tilde{\rho}_1$ and $\tilde{\rho}_2$ into account. $\square$

## Length Trace Formula.

Let $K : [0, \infty) \to \mathbb{R}$ such that $|K(\rho)| \leq \text{const} \cdot e^{-(1+\varepsilon)\rho}$, $\varepsilon > 0$, and $k(z, w) = K(d(z, w))$, $z, w \in \mathbb{H}$.

For a compact hyperbolic surface $M = \Gamma \backslash \mathbb{H}$,
we define: $\overline{k}(z,w) = \sum_{\gamma \in \Gamma} k(z, \gamma w)$.

Since $\#\{\gamma \in \Gamma : d(z, \gamma w) < T\} \leq \text{const} \cdot e^T$,
the sum converges and defines a kernel on $M$.
We consider the integral operator:

$$\mathcal{K}_M : f \longmapsto \int_M \overline{k}(z,w) f(w)\, d\mu(w).$$

<u>Thm</u> (length trace formula)
Assume that $\Gamma \setminus \{e\}$ consists of hyperbolic elements.
Then:

$$\mathrm{Tr}(K_M) = \int_M \overline{k}(z,z)\, d\mu(z) =$$

$$= |M| \cdot K(0) + \sum_{n \geq 1} \sum_{\sigma \in \mathcal{G}(M)} \frac{|\sigma|}{\sqrt{\cosh(n|\sigma|) - 1}} \cdot \int_{n|\sigma|}^{\infty} \frac{K(\rho) \sinh \rho \, d\rho}{\sqrt{\cosh \rho - \cosh(n|\sigma|)}}.$$

Let $\Pi$ be a set of representatives of conjugacy classes of primitive elements in $\Gamma$.

Then $\forall \gamma \in \Gamma \setminus \{e\}$: $\gamma = r^{-1} s^n r$ for $r \in \Gamma$, $s \in \Pi$, $n \in \mathbb{N}$, where $s$ and $n$ are uniquely determined.

We note that $r_1^{-1} s r_1 = r_2^{-1} s r_2 \Leftrightarrow r_1 r_2^{-1} \in Z_s$, where $Z_s$ = the centraliser of $s$ in $\Gamma$.

Let $R_s$ be a set of representatives for cosets of $Z_s \backslash \Gamma$.

Then $\Gamma \setminus \{e\} = \bigsqcup_{n \geq 1,\, s \in \Pi,\, r \in R_s} \{r^{-1} \cdot s^n \cdot r\}$, and

$$\int_M \bar{k}(z,z)\, d\mu(z) \underset{\uparrow}{=} \int_F \Big( \sum_{\gamma \in \Gamma} k(z, \gamma z) \Big) dm(z)$$

F = a compact fundamental domain

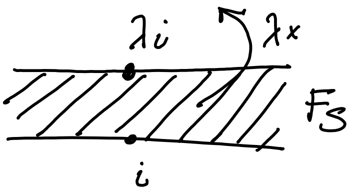
$$= |F| \cdot K(0) + \sum_{n \geq 1,\, s \in \Pi} \Big( \sum_{r \in R_s} \int_F k(z, r^{-1} s^n r z)\, dm(z) \Big).$$

$$\sum_{r \in R_s} \int_F k(z, r^{-1} s^n r z)\, dm(z) = \sum_{r \in R_s} \int_F k(rz, s^n r z)\, dm(z)$$

$$= \int_{R_s F} k(z, s^n z)\, dm(z).$$

We note that $R_s F$ is a fundamental domain for $Z_s$, and $\int_{R_s F} k(z, s^n z)\, dm(z) = \int_{F_s} k(z, s^n z)\, dm(z)$ for <u>any</u> fundamental domain $F_s$ of $Z_s$.

Since $m$ is $SL_2(\mathbb{R})$-inv., we may replace $S$ by its conjugate and assume that




$$Sz=\lambda z, \quad \lambda>1,$$

$$F_S=\{z\in\mathbb{H}:\ 1\le \mathrm{Im}(z)<\lambda\}.$$

Then
$$\int_{F_S} k(z,\lambda^n z)\,dm(z) = \int_1^{\lambda}\int_{\mathbb{R}} K(d(z,\lambda^n z))\,\frac{dx\,dy}{y^2}$$

$$= \int_1^{\lambda}\int_{\mathbb{R}} K\Big(d\Big(i, \frac{(\lambda^n-1)x}{y}+i\lambda^n\Big)\Big)\,\frac{dx\,dy}{y^2}$$

$z\mapsto \frac{z-x}{y}$, $\lambda^n z\mapsto \frac{\lambda^n z-x}{y}$ — isometry

$$= \int_1^{\lambda}\int_{\mathbb{R}} K(d(i, a+i\lambda^n))\,\frac{dy}{y}\,da$$

$a=\frac{(\lambda^n-1)x}{y}$

$$= \frac{\log\lambda}{\lambda^n-1}\cdot\int_{\mathbb{R}} K(d(i, a+i\lambda^n))\,da.$$

This is "horospherical transform", which we have computed previously (see (**)):

$$\int_{\mathbb{R}} K(d(i, a+i\lambda^n))\,da = \sqrt{2}\cdot\lambda^{n/2}\int_{\log(\lambda^n)} \frac{K(\rho)\sinh\rho\,d\rho}{\sqrt{\cosh\rho - \cosh(\log(\lambda^n))}}.$$

We note that $\lambda = e^{|\sigma|}$ where $|\sigma|$ is the length of the corresponding periodic geodesic. Hence,

$$\int_{F_S} k(z, S^n z)\,dm(z) = \frac{2^{-1/2}\cdot|\sigma|}{|\sinh(n|\sigma|/2)|}\cdot\int_{n|\sigma|}^{\infty}\frac{K(\rho)\sinh\rho\,d\rho}{\sqrt{\cosh\rho - \cosh(n|\sigma|)}}.$$

$\lrcorner$

<u>Thm.</u> (spectral trace formula)
Let $P_t$ be the integral operator defined by the heat kernel. Then

$$\mathrm{Tr}(P_t) = \sum_{\lambda\in \mathrm{Spec}(\Delta)} e^{-\lambda t}.$$

$\ulcorner$ Let $\{\varphi_i\}$ be the orthonormal eigenbasis for $\Delta$.
Then $\overline{p_t}(z,\cdot) = \sum_i c_i(t,z)\cdot\varphi_i$, where
$c_i(t,z) = \langle \overline{p_t}(z,\cdot), \varphi_i\rangle = P_t\varphi_i$ is the solution

of the heat equation with the initial condition $f=\varphi_i$, so that $c_i(t,z)=e^{-\lambda_i t}\varphi_i$, and $\overline{P}_t(z,w)=\sum_{i\geq 0} e^{-\lambda_i t}\varphi_i(z)\varphi_i(w)$.

One can show that the convergence is uniform.

Hence, $\mathrm{Tr}(P_t)=\int_M \overline{P}_t(z,z)\,d\mu(z)=\sum_i e^{-\lambda_i t}\int_M \varphi_i^2\,d\mu$

$$= \sum_i e^{-\lambda_i t}.$$

$\square$

<u>Cor</u> (heat trace formula)

$$\sum_{\lambda\in \mathrm{Spec}(\Delta)} e^{-\lambda t} = |M|\cdot(4\pi t)^{-3/2}\cdot e^{-t/4}\int_0^\infty \frac{r\cdot e^{-r^2/4t}}{\sinh(r/2)}\,dr$$

$$+\frac{1}{2}(4\pi t)^{-1/2}e^{-t/4}\cdot\sum_{n\geq 1}\sum_{\sigma\in \mathcal{G}(M)}\frac{|\sigma|}{\sinh\left(\frac{n|\sigma|}{2}\right)}\cdot e^{-\frac{n^2|\sigma|^2}{4t}}.$$

Combine the previous theorems and the formula for the inverse Abel transform.

$\square$

**Cor** (Weyl law)

$$N(T) = \#\{\lambda \in \operatorname{Spec}(\Delta) : \lambda < T\} \sim \frac{|M|}{4\pi} \cdot T \quad \text{as } T \to \infty.$$

The trace formula gives the asymptotics for the Laplace transform $\int_0^\infty e^{-tx}\, dN(x)$ as $t \to 0^+$.

We obtain:

$$\text{"1st term"} \sim \frac{|M|}{4\pi} \cdot t^{-1} \quad \text{as } t \to 0^+$$

$$\text{"2nd term"} \longrightarrow 0 \quad \text{as } t \to 0^+.$$

Now the statement follows from the Tauberian Thm. $\square$

# Lecture 7.

## Microlocal lifts and quantum ergodicity.

$M = \Gamma \backslash \mathbb{H}$ – compact hyperbolic surface.

$\Delta : C^\infty(M) \to C^\infty(M)$ – Laplace operator

$\{\varphi_n\}$ – orthonormal basis of eigenfunctions of $\Delta$ with eigenvalues $\lambda_i \to \infty$.

Consider the sequence of prob. measures:

$$\mu_n = |\varphi_n|^2 d\mu \quad \text{on } M.$$

$\mu_n$ is the "distribution" of the position of the particle with energy $\lambda$.

Conj. (Quantum unique ergodicity / Rudnick–Sarnak)

$$\boxed{\mu_n \xrightarrow[n\to\infty]{} \text{Area}}$$

i.e., $\forall f \in C(M)$: $\int_M f \, d\mu_n \to \int_M f \, d\mu$.

---

$\mu_n \xrightarrow[\text{Microlocal lift}]{} \nu_n$ = prob. measures on $T^1M \simeq \Gamma \backslash SL_2(\mathbb{R})$:

- $\nu_n$ project to $\mu_n$,
- limits of $\nu_n$ are invariant under the geodesic flow.

# Differential operators.

$G = SL_2(\mathbb{R})$

$\mathfrak{g} = \{X \in M_2(\mathbb{R}) : \exp(tX) \in G \text{ for all } t \in \mathbb{R}\}$,

where $\exp(Y) = \sum_{i=0}^{\infty} \frac{Y^i}{i!}$ is the exponential map.

Lie algebra of $G$.

ex. Show that $\mathfrak{g} = \{X : Tr(X) = 0\}$.

Def For $X \in \mathfrak{g}$, we define differential operator:

$$D_X : C^\infty(G) \to C^\infty(G) : f \mapsto \frac{d}{dt} f(g \cdot \exp(tX))\Big|_{t=0}.$$

Properties:

1) $D_{\alpha X + \beta Y} = \alpha D_X + \beta D_Y$,
2) $\pi(g) D_X \pi(g^{-1}) = D_{gXg^{-1}}$,
3) $D_X^* = D_{-X}$,
4) $D_X D_Y - D_Y D_X = D_{[X,Y]}$,
   where $[X,Y] = XY - YX \in \mathfrak{g}$.

<u>Proof (4):</u> $D_X D_Y f(g) = \frac{\partial^2}{\partial t_1 \partial t_2} f(g\exp(t_2 X)\exp(t_1 Y))\Big|_{t_1=t_2=0}$.

$$\exp(t_2 X)\exp(t_1 Y) = \exp(t_1 Y)\cdot \exp(t_2 \cdot \underbrace{\exp(t_1 Y)^{-1} X \exp(t_1 Y)}_{X_{t_1}}),$$

where $X_{t_1} = (I - tY + \dots) X (I + tY + \dots) = I + t\cdot[X,Y] + O(t^2)$.

Hence, by the chain rule,

$$D_X D_Y f(g) = \frac{\partial^2}{\partial t_1 \partial t_2} f(g\exp(t_1 Y)\exp(t_2 X_{t_1}))\Big|_{t_1=t_2=0}$$

$$= \frac{d}{dt_1} D_{X_{t_1}} f(g\exp(t_1 Y))\Big|_{t_1=0}$$

$$= D_{[X,Y]} f + D_Y D_X f. \qquad \square$$

We also define $D_X$ for $X \in \mathfrak{g} \otimes \mathbb{C}$ by linearity.

<u>Notation:</u> $H = \begin{pmatrix} 1/2 & 0 \\ 0 & -1/2 \end{pmatrix} \leftarrow$ direction of the geodesic flow,

$U^+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, $U^- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \leftarrow$ direction of horocycle flows,

$W = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \leftarrow$ direction of $K = SO(2)$.

$\Omega = D_H D_H + \frac{1}{2} D_{V+} D_{V-} + \frac{1}{2} D_{V-} D_{V+}$

Casimir operator

ex. 1) $\Omega$ - commutes with all other $D_X$,

2) $\Omega|_{C^\infty(G)^K} = \Delta$, where we identify $G/K \simeq \mathbb{H}$.

## Fourier Expansion.

Let $k_\theta = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \in K = SO(2)$.

For $f \in C^\infty(G)$, define

$$f_n(g) = \int_K f(g k_\theta) \cdot e^{-in\theta} \frac{d\theta}{2\pi}.$$

Then $f_n(g k_\sigma) = f_n(g) \cdot e^{in\sigma}$, and

$$\boxed{f(g) = \sum_{n \in \mathbb{Z}} f_n(g)}$$

Let $\mathcal{A}_n = \{f : f(g k_\theta) = f(g) \cdot e^{in\theta}\}$.

Since $D_W$ is the derivative along $k_\theta$,

$$\mathcal{A}_n = \{f : D_W f = in f\}.$$

<u>Def.</u> $f$ is <u>K-finite</u> if it lies in a span of finitely many $\mathcal{A}_n$'s.

The linear map $X \mapsto [W, X]$ can be diagonalised: for $E^+ = \frac{1}{2}\begin{pmatrix} 1 & i \\ i & -1 \end{pmatrix}$ and $E^- = \frac{1}{2}\begin{pmatrix} 1 & -i \\ -i & -1 \end{pmatrix}$,

$$[W, E^\pm] = \pm 2i\, E^\pm.$$

This implies that

$$D_{E^\pm}(\mathcal{A}_n) \subset \mathcal{A}_{n\pm 2}.$$

Indeed, for $f \in \mathcal{A}_n$,

$$D_W D_{E^+} f = D_{E^+} D_W f + D_{[W, E^+]} f = (n+2) i \cdot D_{E^+} f.$$

$\{E^+, E^-, W\}$ forms a basis of $\mathfrak{g} \otimes \mathbb{C}$, and

$$\Omega = D_{E^+} D_{E^-} - \frac{1}{4} D_W^2 + \frac{i}{2} D_W$$
$$= D_{E^-} D_{E^+} - \frac{1}{4} D_W^2 - \frac{i}{2} D_W.$$

# Microlocal lift.

Let $\varphi \in C^\infty(\Gamma\backslash\mathbb{H})$ be an eigenfunction of $\Delta$ with eigenvalue $\lambda = \frac{1}{4} + r^2$, $\|\varphi\|_2 = 1$.

We aim to construct a prob. measure on $\Gamma\backslash G$ which projects to $|\varphi|^2 d\mu$ and is assymptotically invariant under the geodesic flow as $\lambda \to \infty$.

First, we construct a "distribution" $I_\varphi$.

We define inductively:

$$\varphi_0(g) = \varphi(gK),$$

$$\varphi_{2n+2} = \frac{1}{ir+\frac{1}{2}+n} D_{E^+} \varphi_{2n}, \quad n \geq 0,$$

$$\varphi_{2n-2} = \frac{1}{ir+\frac{1}{2}-n} D_{E^-} \varphi_{2n}, \quad n \leq 0.$$

Since $\Omega$ commutes with $D_{E^\pm}$, $\Omega\varphi_{2n} = \lambda\varphi_{2n}$.

Then $\|D_{E^+}\varphi_{2n}\|^2 = \langle D_{E^+}^* D_{E^+}\varphi_{2n}, \varphi_{2n}\rangle = -\langle D_{E^-} D_{E^+}\varphi_{2n}, \varphi_{2n}\rangle$

$$= -\langle (\Omega + \tfrac{1}{4}D_W^2 + \tfrac{i}{2}D_W)\varphi_{2n}, \varphi_{2n}\rangle$$

$$= (-\lambda + \tfrac{1}{4}n^2 + \tfrac{1}{2}n)\|\varphi_{2n}\|^2 = |ir + \tfrac{1}{2} + \tfrac{1}{2}n|^2 \cdot \|\varphi_{2n}\|^2.$$

Hence, $\|\varphi_n\|_2 = 1$.

For $K$-finite $f \in C^\infty(\Gamma\backslash G)$, we define

$$I_\varphi(f) = \langle f \cdot \underbrace{\sum_{n\in\mathbb{Z}} \varphi_{2n}}_{\varphi_\infty}, \varphi_0 \rangle.$$

Since $f$ is $K$-finite, $\langle f\varphi_{2n}, \varphi_0\rangle = 0$ for all but finitely many $n$.

Clearly, for $f \in \mathcal{A}_0$,

$$\boxed{I_\varphi(f) = \langle f, |\varphi|^2 \rangle = \int_M f \, d\mu_\lambda .}$$

In fact, $I_\varphi$ is asymptotically a measure:

Lem. 1. Let $\psi = \frac{1}{\sqrt{2N+1}} \cdot \sum_{n=-N}^{N} \varphi_{2n}$ with $N = [\sqrt{r}]$,

$\nu = |\psi|^2 dm$ – prob. measure on $\Gamma\backslash G$.

Then for $K$-finite $f \in C^\infty(\Gamma\backslash G)$,

$$\boxed{I_\varphi(f) = \int_{\Gamma\backslash G} f \, d\nu + O_f(r^{-1/2}).} \qquad (*)$$

We have $\int_{\Gamma\backslash G} f\, d\nu = \langle f\psi, \psi\rangle = \frac{1}{2N+1} \cdot \sum_{n,m=-N}^{N} \langle f\varphi_{2n}, \varphi_{2m}\rangle$.

Suppose that $f \in \sum_{\ell=-2L}^{2L} \mathcal{A}_\ell$. Then

$$\langle f\varphi_{2n}, \varphi_{2m}\rangle = 0 \quad \text{for } |n-m| > L$$

because the spaces $\mathcal{A}_\ell$ are orthogonal.

For $|n-m| \leq L$,

$$\langle f\varphi_{2n}, \varphi_{2m}\rangle = \frac{1}{ir+\frac{1}{2}+n-1} \langle f D_{E^+}\varphi_{2n-2}, \varphi_{2m}\rangle$$

$$= \frac{1}{ir-\frac{1}{2}+n} \cdot \left[\langle D_{E^+}(f\varphi_{2n-2}), \varphi_{2m}\rangle - \underbrace{\langle D_{E^-}(f)\varphi_{2n-2}, \varphi_{2m}\rangle}_{\text{Bounded}}\right]$$

$$= \frac{1}{ir-\frac{1}{2}+n} \cdot \langle f\varphi_{2n-2}, \underset{\substack{\shortparallel \\ -D_{E^-}}}{D^*_{E^+}}\varphi_{2m}\rangle + O(r^{-1})$$

$$= -\underbrace{\frac{-ir+\frac{1}{2}-m}{ir-\frac{1}{2}+n}}_{1+O\left(\frac{|m-n|}{r}\right)} \langle f\varphi_{2n-2}, \varphi_{2m-2}\rangle + O(r^{-1})$$

$$= \langle f\varphi_{2n-2}, \varphi_{2m-2}\rangle + O(r^{-1}).$$

$$= \langle f\varphi_{2(n-m)}, \varphi_0\rangle + O(\underbrace{Nr^{-1}}_{r^{-1/2}})$$

Hence, $\langle f\psi, \psi\rangle = \frac{1}{2N+1} \cdot \sum_{\substack{m=n=-N \\ |m-n|\leq L}}^{N} \left(\langle f\varphi_{2(n-m)}, \varphi_0\rangle + O(r^{-1/2})\right)$

$$= \sum_{\ell=-L}^{L} \underbrace{\frac{2N+1-|\ell|}{2N+1}}_{1+O(N^{-1})} \left(\langle f\varphi_{2\ell}, \varphi_0\rangle + O(r^{-1/2})\right)$$

$$= \underbrace{\left\langle f\cdot \sum_{\ell=-L}^{L} \varphi_{2\ell}, \varphi_0 \right\rangle}_{I_\varphi(f).} + O(r^{-1/2})$$

$\underline{\text{Lem. 2}}$ $\exists$ fixed differential operator $\mathcal{L}$:

$\forall K$-finite $f \in C^\infty(\Gamma\backslash G)$: $\boxed{I_\varphi((rD_H + \mathcal{L})f) = 0.}$ $(**)$

Recall that $I_\varphi(f) = \langle f \underbrace{\varphi_\infty}_{\sum_{n\in\mathbb{Z}} \varphi_{2n}}, \varphi_0\rangle$.

We obtain:

$$\lambda\langle f\varphi_\infty, \varphi_0\rangle = \langle f\varphi_\infty, \Omega\varphi_0\rangle = \langle f\varphi_\infty, D_{E^-}D_{E^+}\varphi_0\rangle$$

$$= \langle \underbrace{(D_{E^-}D_{E^+})^*}_{D_{E^-}D_{E^+}}(f\varphi_\infty), \varphi_0\rangle = \langle D_{E^-}D_{E^+}(f)\cdot\varphi_\infty, \varphi_0\rangle +$$

$$+\langle D_{E^+}(f)D_{E^-}(\varphi_\infty),\varphi_0\rangle+\langle D_{E^-}(f)D_{E^+}(\varphi_\infty),\varphi_0\rangle+\langle fD_{E^-}D_{E^+}(\varphi_\infty),\varphi_0\rangle,$$

where

$$D_{E^-}(\varphi_\infty)=\sum_{n\in\mathbb{Z}}(ir+\tfrac{1}{2}-n)\underbrace{\varphi_{2n-2}}_{\substack{\cap\\ a_{2n-2}}}=(ir+\tfrac{i}{2}D_W-\tfrac{1}{2})\varphi_\infty,$$

$$D_{E^+}(\varphi_\infty)=\sum_{n\in\mathbb{Z}}(ir+\tfrac{1}{2}+n)\varphi_{2n+2}=(ir-\tfrac{i}{2}D_W-\tfrac{1}{2})\varphi_\infty,$$

$$D_{E^-}D_{E^+}(\varphi_\infty)=(\Omega-\tfrac{1}{4}D_W^2-\tfrac{i}{2}D_W)\varphi_\infty=\lambda\varphi_\infty-(\tfrac{1}{4}D_W^2+\tfrac{i}{2}D_W)\varphi_\infty.$$

Then $\lambda$ cancels out, and the equality becomes:

$$ir\big(\underbrace{\langle D_{E^+}(f)\varphi_\infty,\varphi_0\rangle+\langle D_{E^-}(f)\varphi_\infty,\varphi_0\rangle}_{2i\,I_\varphi(D_H f)}\big)+\big(\underbrace{-\!\!\shortparallel\!\!-}_{\text{independent of } r}\big)=0$$

We note that $(-\!\shortparallel\!-)$ contains only the diff. operator $D_W$. Since $D_W(\varphi_0)=0$,

$$0=-\langle D_W(f_1f_2),\varphi_0\rangle=-\langle D_W(f_1)f_2,\varphi_0\rangle-\langle f_1D_W(f_2),\varphi_0\rangle,$$

and $\langle f_1 \mathcal{D}_W(f_2), \varphi_0 \rangle = -\langle \mathcal{D}_W(f_1) f_2, \varphi_0 \rangle$.

Using this identity, we obtain

$$(\text{—}\text{''}\text{—}) = \langle \mathcal{L}(f) \varphi_\infty, \varphi_0 \rangle = I_\varphi(\mathcal{L}(f)).$$

where $\mathcal{L}$ is an explicit diff. operator.

Combining $(*)$ & $(**)$,

$$\int_{\Gamma\backslash G} \mathcal{D}_H(f)\, d\nu = \int_{\Gamma\backslash G} f\, d\nu + O(r^{-1/2}).$$

Hence, limits of $\nu$ as $\lambda \to \infty$ are invariant under the geodesic flow.

## Quantum Ergodicity.

**Thm.** For every $K$-finite $f \in C^\infty(\Gamma\backslash G)$,

$$\underbrace{\frac{1}{\#\{\lambda \in \mathrm{Spec}(\Delta) : \lambda \le L\}}}_{N(L)} \cdot \sum_{\lambda \in \mathrm{Spec}(\Delta) : \lambda \le L} \left| \int_{\Gamma\backslash G} f\, d\nu_\lambda - \int_{\Gamma\backslash G} f \right|^2 \xrightarrow[L\to\infty]{} 0,$$

where $d\nu_\lambda(g) = |\psi_\lambda(g)|^2 dg$.

In the proof, we use:

<u>General Weyl Law</u>: For every $K$-finite $f$,

$$\sum_{\lambda \in Spec(\Delta):\, \lambda \leq L} \langle f, |\psi_\lambda|^2 \rangle \sim \left( \int_{\Gamma \backslash G} f \right) \cdot \frac{vol(G/\Gamma)}{4\pi} \cdot L .$$

We may assume that $\int_{\Gamma \backslash G} f = 0$.

Let $A_T(f) = \frac{1}{T} \int_0^T \pi(a_t) f \, dt$.
$\uparrow$ geodesic flow

By Lemma 1-2 and the Mean Value Thm:

$$\langle \pi(a_t) f, |\psi_\lambda|^2 \rangle = \langle f, |\psi_\lambda|^2 \rangle + O(t \cdot \lambda^{-1/4}),$$

$$\langle A_T(f), |\psi_\lambda|^2 \rangle = \langle f, |\psi_\lambda|^2 \rangle + O(T \lambda^{-1/4}).$$

Hence,

$$\sum_{L_0 < \lambda \leq L} |\langle f, |\psi_\lambda|^2 \rangle|^2 = \sum_{L_0 < \lambda \leq L} |\langle A_T(f), |\psi_\lambda|^2 \rangle|^2 + O_T(L \cdot L_0^{-1/4})$$

$$\leq \sum_{L_0 < \lambda \leq L} \langle |A_T(f)|^2, |\psi_\lambda|^2 \rangle + O_T(L \cdot L_0^{-1/4}),$$

$\uparrow$ Cauchy-Schwarz inequality

and by the Weyl law,

$$\overline{\lim_{L\to\infty}} \frac{1}{N(L)} \cdot \sum_{\lambda \le L} \left| \langle f, |\psi_\lambda|^2 \rangle \right|^2 \le \| A_T(f) \|_2^2 .$$

Finally, by ergodicity (or mixing) of the geodesic flow, $\| A_T(f) \|_2 \xrightarrow[T\to\infty]{} 0$.

# Lecture 8

## Hecke operators and recurrence of eigenstates.

For $r \in \mathbb{Q}$, write $r = p^n \cdot \frac{\ell}{s}$ with $\ell, s$ coprime to $p$.

Define *p-adic norm*: $|r|_p = p^{-n}$.

Basic properties: $|r_1 \cdot r_2|_p = |r_1|_p \cdot |r_2|_p$,
$|r_1 + r_2|_p \leq \max\{|r_1|_p, |r_2|_p\}$.

$\mathbb{Q}_p$ = completion of $\mathbb{Q}$ with respect to $|\cdot|_p$.
← the field of p-adic numbers
$\cup$
$\mathbb{Z}_p = \{x : |x|_p \leq 1\}$ – compact open subring.

ex. $\mathbb{Z}[\frac{1}{p}] \xhookrightarrow[\text{diag}]{} \mathbb{R} \times \mathbb{Q}_p$ is discrete.

---

Notation: $G = PGL_2(\mathbb{R})$, $\Gamma = PGL_2(\mathbb{Z})$, $X = \Gamma \backslash G$,
$G_p = PGL_2(\mathbb{Q}_p)$, $K_p = PGL_2(\mathbb{Z}_p)$, $\Gamma_p = PGL_2(\mathbb{Z}[\frac{1}{p}])$.
← discrete subgroup of $G \times G_p$.

Prop. $X \simeq \Gamma_p \backslash (G \times G_p) / K_p$ as $G$-spaces.

We claim that the action of $G$ on $\Gamma_p \backslash (G \times G_p) / K_p$ is transitive. Equivalently, $G_p = \Gamma_p \cdot K_p$.

Given $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in G_p$, we show that

$$\exists \gamma \in \Gamma_p,\ k \in K_p : \ \gamma g k = I :$$

$$g \xrightarrow[\times \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}]{} \{ |a|_p \geq |b|_p \} \xrightarrow[\times \begin{pmatrix} 1 & -\frac{b}{a} \\ 0 & 1 \end{pmatrix}]{} \begin{pmatrix} a & 0 \\ c & d' \end{pmatrix} \xrightarrow[\begin{pmatrix} 1 & 0 \\ \alpha & 1 \end{pmatrix} \times]{} \begin{pmatrix} a & 0 \\ c' & d' \end{pmatrix}$$

If $\alpha \in \mathbb{Z}[\frac{1}{p}]$, $\alpha \approx -\frac{c}{a}$, then $c' \approx 0$ and $|c'|_p \leq |d'|_p$.

$$\begin{pmatrix} a & 0 \\ c' & d' \end{pmatrix} \xrightarrow[\times \begin{pmatrix} 1 & 0 \\ -\frac{c'}{d'} & 1 \end{pmatrix}]{} \begin{pmatrix} a & 0 \\ 0 & d' \end{pmatrix}, \quad a, d' \in p^{\mathbb{Z}} \cdot \mathbb{Z}_p^{\times}.$$

This proves the claim.

Now $X \simeq \mathrm{Stab}_G(\Gamma_p (e,e) K_p) \backslash G$.

$$\Gamma_p (g,e) K_p = \Gamma (e,e) K_p \iff \exists \gamma \in \Gamma_p,\ k \in K_p : \begin{cases} \gamma g = e \\ \gamma k = e \end{cases}$$

$$\iff g \in \Gamma_p \cap K_p = \Gamma.$$

The group $G_p$ is equipped with invariant measure $m_p$, which we normalise, so that $m_p(K_p)=1$.

<u>Def</u> For $f: X \to \mathbb{C}$, the <u>Hecke operator</u>

$$T_{p^n} f(\Gamma g) = \int_{B_n} f(\Gamma_p(g,b)K_p)\, dm_p(b),$$

where $B_n = K_p \begin{pmatrix} 1 & 0 \\ 0 & p^n \end{pmatrix} K_p$.

<u>Properties</u>: 1) $T_{p^n}$ is self-adjoint,

2) $T_{p^n}$ commutes with the action of $G$ and $D_X$, $X \in \mathfrak{g}$.

---

A <u>lattice</u> in $\mathbb{Q}_p^2$ is $\mathbb{Z}_p v_1 + \mathbb{Z}_p v_2$ where $\{v_1, v_2\}$ is a basis of $\mathbb{Q}_p^2$.

$L_1 \sim L_2$ if $L_1 = \alpha L_2$ with $\alpha \in \mathbb{Q}_p^\times$.

$X_p = \{$equivalence classes of lattices in $\mathbb{Q}_p^2\}$.

Note that $G_p = PGL_2(\mathbb{Q}_p)$ acts transitively on $X_p$, and $\mathrm{Stab}_{G_p}(\mathbb{Z}_p e_1 + \mathbb{Z}_p e_2) = PGL_2(\mathbb{Z}_p) = K_p$. Hence,

$$X_p \simeq G_p / K_p .$$

<u>Lem.</u> (Cartan decomposition) $GL_2(\mathbb{Q}_p) = GL_2(\mathbb{Z}_p) \cdot \left\{ \begin{pmatrix} p^m & 0 \\ 0 & p^n \end{pmatrix} : m \le n \right\} GL_2(\mathbb{Z}_p)$.

Similar to the previous proof: apply rows/column operations to reduce $g \in G_p$ to diagonal form.

<u>Cor.</u> Let $L$ and $M$ be lattices in $\mathbb{Q}_p^2$. Then $\exists$ a basis $\{v_1, v_2\}$ of $L$ and integers $m \le n$ such that $\{p^m \cdot v_1, p^n \cdot v_2\}$ is a basis of $M$.

Without loss of generality,
$L = \mathbb{Z}_p e_1 + \mathbb{Z}_p e_2$ and $M = g(\mathbb{Z}_p e_1 + \mathbb{Z}_p e_2)$ with $g \in GL_2(\mathbb{Q}_p)$.

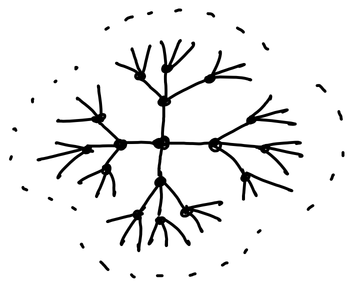
Write $g = k_1 \begin{pmatrix} p^m & 0 \\ 0 & p^n \end{pmatrix} k_2$ with $k_i \in GL_2(\mathbb{Z}_p)$.

Then $\{k_1^{-1} e_1, k_1^{-1} e_2\}$ and $\{k_2 e_1, k_2 e_2\}$ are the required bases.

**Def.** $[L]\neq[M]\in X_p$ are neighbours if for some representatives $L, M$, $pL\subset M\subset L$.

This defines a graph structure on $X_p$.

**Prop.** $X_p$ is a $(p+1)$-regular tree.



Every neighbour of $[L]$ corresponds to a 1-dim. subspace of

$$L/pL \simeq \mathbb{F}_p\oplus\mathbb{F}_p.$$

Hence, there are $p+1$ neighbours.

The claim that $\nexists$ no loops follows from the following lemma.

**Lem.** Consider a chain of lattices

$L_0 \supsetneq L_1 \supsetneq \dots \supsetneq L_n$ such that $L_{i+1}\supsetneq pL_i$, $pL_{i-1}\neq L_{i+1}$.

Then there exists a basis $\{v_1, v_2\}$ of $L_0$ such that $L_i = \mathbb{Z}_p v_1 + \mathbb{Z}_p (p^i v_2)$.

We argue by induction on $n$.

Suppose that $L_i = \mathbb{Z}_p v_1 + \mathbb{Z}_p(p^i v_2)$ for $i<n$.

The lattice $L_n$ corresponds to a 1-dim subspace of $L_{n-1}/pL_{n-1} \simeq \mathbb{F}_p \oplus \mathbb{F}_p$.

Since $L_n \neq pL_{n-2} = \mathbb{Z}_p(pv_1) + \mathbb{Z}_p(p^{n-1}v_2)$, this subspace is not $\langle (0,1) \rangle$.

Hence, $L_n = \underbrace{\langle v_1 + x\cdot p^{n-1} v_2,\ pL_{n-1} \rangle}_{\langle v_1 + x\cdot p^{n-1} v_2,\ p^n v_2 \rangle}$ for some $x = 0, \dots p-1$.

Then $\{ v_1 + xp^{n-1} v_2, v_2 \}$ is a required basis.

In particular, it follows that if $d([L], [\mathbb{Z}_p e_1 + \mathbb{Z}_p e_2]) = n$, then $[L] = \left[ k \begin{pmatrix} 1 & 0 \\ 0 & p^n \end{pmatrix} (\mathbb{Z}_p e_1 + \mathbb{Z}_p e_2) \right]$ for $k \in K_p$.

<u>Notation:</u> For $x = \Gamma g \in X$,

$\Delta_p(x) = \Gamma_p \backslash (\{g\} \times X_p)$ – <u>Hecke tree</u> through $x$

$\cup$

$\Delta_p^{(n)}(x)$ – vertices with distance $n$ from $x$.

Then
$$\boxed{T_{p^n} f(x) = \sum_{y \in \Delta_p^{(n)}(x)} f(y)}$$

This formula implies that:

<u>Lem.</u> 1) $T_p^2 = T_{p^2} + (p+1)I$,

2) $T_p \cdot T_{p^n} = T_{p^{n+1}} + p\, T_{p^{n-1}}$, $n \geq 2$.

<u>Thm.</u> Let $f$ be a function on a tree such that $T_p f = \lambda f$. Then $\exists c > 0$ (absolute constant):
$$\sum_{y:\, d(x,y) \leq n} |f(y)|^2 \geq c \cdot n\, |f(x)|^2.$$

By Cauchy-Schwarz inequality,
$$\left|\sum_{i=0}^{n} T_{p^i} f(x)\right| = \left|\sum_{y:\, d(x,y)\leq n} f(y)\right| \leq \left(\sum_{y:\, d(x,y)\leq n} |f(y)|^2\right)^{1/2} \cdot \#\{y : d(x,y) \leq n\}^{1/2}$$
$$\leq \text{const} \cdot p^{n/2} \cdot \left(\sum_{y:\, d(x,y)\leq n} |f(y)|^2\right)^{1/2}$$

It follows from the lemma that $f$ is also an eigenvector of $T_{p^i}$ and the eigenvalues $\lambda_i$ satisfy the 2nd order recurrence relation:

$$\lambda_{i+1} = \lambda \cdot \lambda_i - p \cdot \lambda_{i-1}, \quad i \geq 2,$$
$$\lambda_1 = \lambda, \quad \lambda_2 = \lambda^2 - p - 1.$$

Hence, $\lambda_i = \alpha \cdot u^i + \beta \cdot v^i$
where $u$ & $v$ are solutions of $x^2 - \lambda x + p = 0$.

It is sufficient to show that

$$\underbrace{\left| \sum_{i=0}^{n} \lambda_i \right|} \geq \text{const} \cdot p^{n/2} \cdot n.$$

This can be computed explicitly as a geometric series.

We omit details...

**Def.** A measure $\nu$ on $X$ is called *Hecke recurrent* if for every measurable $B \subset X$ and $\nu$-almost every $x \in B$,

$$\Delta_p^{(n)}(x) \cap B \neq \emptyset \text{ for infinitely many } n.$$

**Thm.** Let $\varphi_k$ be a sequence of eigenfunctions of $T_p$, $\|\varphi_k\|_2 = 1$, and $d\nu_k = |\varphi_k(g)|^2 d\mu(g)$. Then every limit $\nu$ of $\nu_k$'s is Hecke recurrent.

$\Gamma$ Since $T_{p^i}$ are self-adjoint, it follows from the Theorem that for $f \in C_c(X)$, $f \geq 0$:

$$\left\langle \sum_{i=0}^{n} T_{p^i} f, |\varphi_k|^2 \right\rangle = \left\langle f, \sum_{i=0}^{n} T_{p^i} |\varphi_k|^2 \right\rangle$$

$$\geq \text{const} \cdot n \cdot \langle f, |\varphi_k| \rangle.$$

Passing to the limit, we obtain

$$\int_X \left( \sum_{i=0}^{n} T_{p^i} f \right) d\nu \geq \text{const} \cdot n \cdot \int_X f. \quad (*)$$

This extends to general non-negative measurable functions.

Let $B \subset X$ be a measurable set.

Let $B_k = \{x \in B : B \cap \Delta_\varphi^{(k)}(x) = \emptyset\}$ and $C_\ell = \bigcap_{k \geq \ell} B_k$.

The set of points in $B$ which does not come back to $B$ infinitely often is $\bigcup_{\ell \geq 1} C_\ell$.

We apply $(*)$ to $f = \chi_{C_\ell}$.

For every $z \in X$, $\Delta_\varphi(z) \cap C_\ell$ has diameter $\leq \ell$ on the tree $\Delta_\varphi(z)$. Then $\sum_{i=0}^{\infty} T_{p^i} f$ is bounded, and by $(*)$, $\nu(C_\ell) = 0$.

Hence, for $\nu$-a.e. $x \in B$, $B \cap \Delta_\varphi^{(n)}(x) \neq \emptyset$ for infinitely many $n$. $\square$

# Lecture 9

## Entropy of quantum limits.

$M = \Gamma \backslash \mathbb{H}$ - finite area hyperbolic surface,
$\Gamma$ - arithmetic lattice (e.g., $\Gamma = PGL_2(\mathbb{Z})$).

$\varphi_\lambda$ - eigenfunction of $\Delta$, $\|\varphi_\lambda\|_2 = 1$,
which is also an eigenfunction of $T_p$'s.

$d\mu_\lambda = |\varphi_\lambda|^2 d\mu$ - prob. measure.

Conj (arithmetic quantum unique ergodicity)

$$\boxed{\mu_\lambda \xrightarrow[\lambda\to\infty]{} \mu}$$

This was proved by Lindenstrauss
(modulo divergence issue, for noncompact surfaces)
and Soundararajan (nondivergence).

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The microlocal lift gives measures $d\nu_\lambda = |\psi_\lambda|^2 dm$
on $X = \Gamma \backslash G$ which project to $\mu_\lambda$.

Since $T_p$ commutes with the actions of $G$ and $D_X$, $X \in \mathfrak{g}$, $\psi_\lambda$ is also an eigenfunction of $T_p$.

Let $\nu$ = a limit of $\nu_\lambda$ as $\lambda \to \infty$.

Assume that $\nu$ is a probability measure.

We know: – $\nu$ is $g^t$-invariant,

– $\nu$ is $T_p$-recurrent.

<u>Def.</u> A <u>Bowen ball</u>:

$$B_{n,\delta}(x) = \{ y \in X : \ d(g^i(x), g^i(y)) < \delta \} \quad i = -n, \dots, n.$$

$\nu(B_{n,\delta}(x))$ - small $\iff$ $g^t \curvearrowright (X, \nu)$ is chaotic.

<u>Thm</u> (positive entropy) Bourgain – Lindenstrauss $\exists h > 0$: for $\nu$-a.e. $x$, and infinitely many $n$,

$$\boxed{\nu(B_{n,\delta}) \leqslant c(x,\delta) \cdot e^{-hn},} \quad (E)$$

uniformly on $x$ in compact sets.

Lem. $\exists$ uniform $c>0$: $\forall f \geq 0$:

$$\int_X \left(T_p(f) + T_{p^2}(f)\right) d\nu \geq c \cdot \int_X f \, d\nu .$$

Recall that $T_{p^2} = T_p^2 - p - 1$.

If $T_p \psi = \lambda \psi$, then $T_{p^2} \psi = \lambda_2 \psi$, $\lambda_2 = \lambda^2 - p - 1$.

Hence, either $|\lambda| \geq \frac{1}{2}\sqrt{p}$ or $|\lambda_2| \geq \frac{1}{2} p$.

Suppose, for instance, that $|\lambda| \geq \frac{1}{2}\sqrt{p}$.

Then by Cauchy-Schwartz inequality,

$$\frac{1}{2}\sqrt{p}\, |\psi(x)| \leq |T_p \psi(x)| \leq \Big(\sum_{y \in \Delta_p^{(1)}(x)} |\psi(y)|^2\Big)^{1/2} \cdot \sqrt{p+1} .$$

Similar argument also applies in the other case, and obtain $T_p(|\psi|^2) + T_{p^2}(|\psi|^2) \geq c \cdot |\psi|^2$.

Finally, $\int_X \left(T_p(f) + T_{p^2}(f)\right) d\nu_\lambda = \langle T_p f + T_{p^2} f, |\psi_\lambda|^2 \rangle$

$= \langle f, T_p(|\psi_\lambda|^2) + T_{p^2}(|\psi_\lambda|^2) \rangle \geq c \cdot \int_X f \, d\nu_\lambda$,

and passing to limit as $\lambda \to \infty$, we deduce the lemma.

**Proof of Thm.** 1) Recurrence along Hecke tree.

Let $B_n = B_{n,\delta}(x) = x \cdot \underbrace{\left( \bigcap_{i=-n}^{n} a^i G_\delta a^{-i} \right)}_{G_{n,\delta}}$, where

$a = \begin{pmatrix} e^{1/2} & 0 \\ 0 & e^{-1/2} \end{pmatrix}$ and $G_\delta$ is a $\delta$-nbhd of identity in $G$.

We suppose that $\boxed{\nu(B_n) \geq e^{-hn}}$, $h \approx 0$, $(+)$

for all sufficiently large $n$.

By Lemma, $\displaystyle\int_X \sum_{p \leq Q} \left( T_p(\chi_{B_n}) + T_{p^2}(\chi_{B_n}) \right) d\nu \geq c \cdot \frac{Q}{\log Q} \cdot |B_n|.$

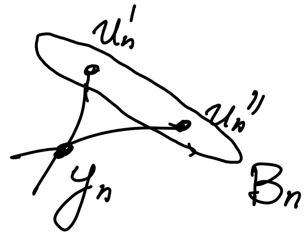
We pick $Q_n \approx e^{2hn}$, so that

$$\int_X \sum_{p \leq Q_n} \left( T_p(\chi_{B_n}) + T_{p^2}(\chi_{B_n}) \right) d\nu > 1,$$

and $\displaystyle\sum_{p \leq Q_n} \left( \sum_{y \in \Delta_p^{(1)}(y_n) \cup \Delta_p^{(2)}(y_n)} \chi_{B_n}(y) \right) > 1$ for some $y_n \in X$.

At least <ins>two</ins> terms in the sum are $\neq 0$.

For simplicity, let's say




$$\exists\, u_n' \neq u_n'' \in \Delta_p^{(1)}(y_n) \cap B_n.$$

Write $y_n = \Gamma g_n$. Then $\Delta_p^{(1)}(y_n) = \Gamma_p(g_n, K_p\begin{pmatrix}1&0\\0&p\end{pmatrix}K_p)/K_p$.

Since $\Gamma = PGL_2(\mathbb{Z})$ is dense in $K_p = PGL_2(\mathbb{Z}_p)$,

the cosets $K_p\begin{pmatrix}1&0\\0&p\end{pmatrix}K_p/K_p$ can be represented by

$\{\gamma_i\} \subset \Gamma\begin{pmatrix}1&0\\0&p\end{pmatrix}\Gamma$, and $\Gamma_p(g_n, K_p\begin{pmatrix}1&0\\0&p\end{pmatrix}K_p)/K_p = \{\Gamma_p(g_n,\gamma_i)K_p\}$

$= \{\Gamma_p(\gamma_i^{-1}g_n, e)K_p\}$.

Hence, $\exists\, \gamma', \gamma'' \in \Gamma\begin{pmatrix}1&0\\0&p\end{pmatrix}\Gamma$: $u_n' = \Gamma(\gamma')^{-1}g_n$, $u_n'' = \Gamma(\gamma'')^{-1}g_n$.

$\gamma'\Gamma \neq \gamma''\Gamma$.

Write $x = \Gamma g$. Then $u_n' = \Gamma g b'$, $u_n'' = \Gamma g b''$

for $b', b'' \in G_{n,\delta}$.

Then for some $\gamma', \gamma'' \in \Gamma\begin{pmatrix}1&0\\0&p\end{pmatrix}\Gamma$, $\gamma'\Gamma \neq \gamma''\Gamma$:

$$\begin{cases}(\gamma')^{-1}g_n = gb', \\ (\gamma'')^{-1}g_n = gb''.\end{cases} \Rightarrow \quad \underbrace{(\gamma')^{-1}\gamma''}_{\eta_n} = g\underbrace{b'(b'')^{-1}}_{b}g^{-1}$$

where $\eta_n \in SL_2(\mathbb{Q})$ with denominator $\leq p$, $\eta_n \neq I$

$$b \in G_{n,\delta}\cdot G_{n,\delta}^{-1} \subset G_{n,2\delta},$$

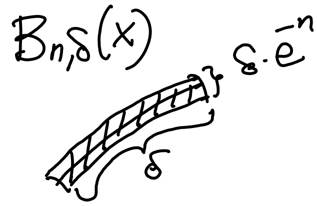
$$\boxed{g^{-1}\eta_n g \in G_{n,2\delta}} \quad (*)$$

## 2) Structure of Bowen balls.

For $b \in G_\delta$, $\quad b = \exp(tH + \alpha U^+ + \beta U^-)$ with $|t|, |\alpha|, |\beta| \ll \delta$,

and
$$a^i \cdot b \cdot a^{-i} = \exp(a^i(tH + \alpha U^+ + \beta U^-)a^{-i})$$
$$= \exp(tH + (\alpha \cdot e^{-i})U^+ + (\beta \cdot e^i)U^-).$$

Hence, $G_{n,\delta} = \bigcap_{i=-n}^{n} a^i G_\delta a^{-i}$ is $(\delta \cdot e^{-n})$-neighbourhood of piece of diagonal.




For $g_1, g_2 \in G_{n,\delta}$, $\quad g_i = a_i v_i$

with diagonal $a_i = I + O(\delta)$, $\quad v_i = I + O(\delta \cdot e^{-n})$.

Then
$$[g_1, g_2] = (a_1 v_1)^{-1}(a_2 v_2)^{-1}(a_1 v_1)(a_2 v_2)$$
$$= v_1^{-1} a_1^{-1} v_2^{-1} a_2^{-1} a_1 v_1 a_2 v_2$$
$$= v_1^{-1} \underbrace{(a_1^{-1} v_2^{-1} a_1)}_{I + O(\delta e^{-n})} \cdot \underbrace{(a_2^{-1} v_1 a_2)}_{I + O(\delta e^{-n})} \cdot v_2$$
$$= I + O(\delta e^{-n}).$$

Now we apply this estimate to $\eta_n$'s.

3) Classification of $\eta_n$'s.

Then $[\eta_n, \eta_{n+1}] = I + O(\delta \cdot e^{-n})$.

On the other hand, $[\eta_n, \eta_{n+1}]$ is rational with denominator $\leq p^4 \leq e^{8hn}$, $h \approx 0$.

This implies that $[\eta_n, \eta_{n+1}] = I$ for sufficiently large $n$.

Now we claim that $\eta_n$'s are hyperbolic ($\Leftrightarrow |\mathrm{Tr}(\eta_n)| > 2$).

Since $g^{-1} \eta_n g = a_n v_n$ with $a_n = \begin{pmatrix} e^{t_n/2} & 0 \\ 0 & e^{-t_n/2} \end{pmatrix}$,
$v_n = I + O(\delta e^{-n})$,

$$\mathrm{Tr}(\eta_n) = e^{t_n/2} + e^{-t_n/2} + O(\delta e^{-n}).$$

If $|\mathrm{Tr}(\eta_n)| \leq 2$, then $e^{t_n/2} + e^{-t_n/2} \leq 2 + O(\delta e^{-n})$,
and $t_n = O(\delta e^{-n})$.

Then $\eta_n = I + O(\delta \cdot e^{-n})$. Since the denominator of $\eta_n$ is $\geq e^{-2hn}$, $h \approx 0$, $\eta_n = I$, which is a contradiction.

Hence, all $\eta_n$'s are hyperbolic and commuting.

In particular, $g^{-1}\eta_n g \in A$ – a fixed diagonilisable subgroup.

Write $g^{-1}\eta_n g = \exp(t_n X)$ with fixed $\|X\| = 1$.

Since $g^{-1}\eta_n g \in G_{\eta,\delta}$, $t_n X = t_n' H + O(e^{-n})$, $t_n' \geq e^{-n/2}$.

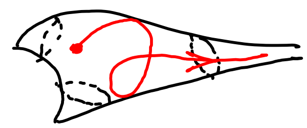
This implies that $X = H$ and $A$ is the diagonal subgroup.

1) Suppose that $\eta = \eta_n$ is <u>diagonalisable over $\mathbb{Q}$.</u>

$\exists h \in SL_2(\mathbb{R})$ proportional to a rational matrix: $h^{-1}\eta h \in A$.

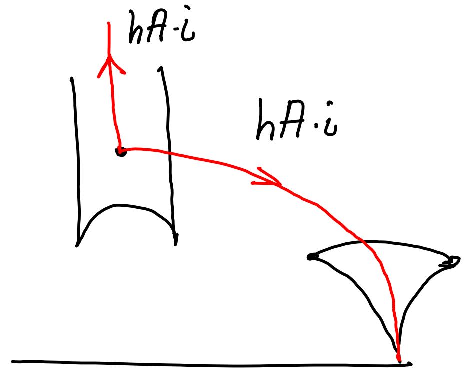
Then $g \in h \cdot N_{SL_2(\mathbb{R})}(A) = hA \cup h\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} A$,

so $x = \Gamma g \in \Gamma h A \cup \Gamma h \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} A$.



The end points of the geodesic $h \cdot A \cdot i$ (and $\Gamma h \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} A$) are rational.

This implies that diverges (i.e., escapes every compact set in $G/\Gamma$).



By Poincare recurrence, $\nu(\Gamma h A)=0$, and $x$ belongs to the set zero measure:

$$\bigcup_h \left(\Gamma h A \cup \Gamma h \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} A\right).$$

2) Suppose that $\eta = \eta_n$ is <u>not diagonalisable</u> $/\mathbb{Q}$ (i.e., the characteristic polynomial of $\eta$ is irreducible). Then $K = \mathbb{Q}[\eta] \subset M_2(\mathbb{Q})$ is a quadratic field. Let $\mathcal{O} = \mathbb{Z}[\eta]$ and $\mathcal{O}^\times$ is the group of invertable elements. Since $\eta \in M_2(\mathbb{Q})$, $\underbrace{\mathcal{O} \cap M_2(\mathbb{Z})}_{\mathcal{U}}$ has finite index in $\mathcal{O}$.

<u>Facts:</u> 1) $\mathcal{U}^\times$ has finite index in $\mathcal{O}^\times$;

2) $\exists$ primes $p$: $\mathcal{U}^\times \cdot p^{\mathbb{Z}}$ has finite index in $\mathcal{O}[\frac{1}{p}]^\times$.

Recall that $\eta \cdot g = g \cdot a$ for some $a \in A$.

By 1), $\eta^k \in \mathcal{U}^\times \subset Gl_2(\mathbb{Z})$ for some $k \in \mathbb{N}$.

Then $\Gamma g a^k = \Gamma \eta^k g = \Gamma g$, i.e. $x = \Gamma g$ is contained in a periodic orbit $xA$.

We claim that $\nu(xA)=0$.

Suppose not. Then we apply p-Hecke recurrence:

$\exists y \in xA: y_n \in \Delta_p(y): y_n \neq y_m$ for $n \neq m$, $y_n \in xA$.

Multiplying by $a \in A$, we may assume that $y=x$.

As in step 1, $y_n = \Gamma \gamma_n^{-1} g$ for $\gamma_n = PGL_2(\mathbb{Z}[\frac{1}{p}])$,

$\gamma_n K_p \neq \gamma_m K_p$ for $n \neq m$.

As above, we have $\gamma_n^{-1} g = g a_n$ for some $a_n \in A$.

Then $\gamma_n^{-1} \in K$ and $\gamma_n^{-1} \in \mathcal{O}[\frac{1}{p}]^{\times}$.

By 2), $\mathcal{U}^{\times} p^{\mathbb{Z}}$ has finite index in $\mathcal{O}[\frac{1}{p}]^{\times}$.

Hence, $\exists n \neq m$: $\mathcal{U}^{\times} p^{\mathbb{Z}} \cdot \gamma_n^{-1} = \mathcal{U}^{\times} p^{\mathbb{Z}} \cdot \gamma_m^{-1}$.

Since $\mathcal{U}^{\times} \subset GL_2(\mathbb{Z})$, this implies that

$$PGL_2(\mathbb{Z}_p)\gamma_n^{-1} \neq PGL_2(\mathbb{Z}_p)\gamma_m^{-1},$$

which is a contradiction. Hence, $\nu(xA)=0$.

Assuming (+), we proved that $x$ belongs either to divergent $A$-orbit or periodic $A$-orbit. The union of such sets have $\nu$-measure $0$. $\blacksquare$

Now the proof of A.Q.U.E. reduces to the problem of classification of measures:

**Thm.** (measure rigidity) Lindenstrauss

Let $\nu$ be a prob. measure on $X$ such that:

1) $\nu$ is invariant under a geodesic flow,
2) $\nu$ is $p$-Hecke recurrent for some $p$,
3) $\nu$ has "positive entropy" (condition $E$).

Then $\nu$ is the invariant measure.